Symmetry properties of conjectural price responses

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***Abstract*—Conjectural variations and conjectural price responses based models are a popular way of analysing oligopolistic electricity markets. The purpose of this paper is to show certain symmetry properties that conjectural price responses have. These properties are useful in order to devise efficient simulation algorithms able to take into account the agents' anticipations of the congestion status of the power grid.**

Keywords: **Power system economics, Interconnected power systems.**

I. Introduction

Competitive electricity markets are nowadays established institutions in a number of countries. However, technical characteristics of electricity as well as historical and policy constraints made the sector prone to oligopolistic behaviour.

Under perfect competition assumptions, detailed and precise predictions of market behaviour can be derived from economic theory. Unfortunately the situation is much less satisfactory in the case of oligopolistic competition. As a consequence, a number of approximate and heuristic approaches to oligopolistic electricity market simulation have been developed, in order to provide insight and even rough forecasts for companies and regulators [1, 2, 3, 4, 5, 6, 7].

Among these models, those based on "conjectural parameters" (conjectural variations, conjectural price responses or conjectural supply functions) stand out as relatively popular among practitioners. The conditional parameters summarize the assumptions about the agents' strategies. Besides, mathematically the models resemble others traditionally used in the electricity industry or in other sectors [8, 9, 10, 11, 12].

However, the application of these methods requires sensible values for the conjectural parameters. The task is specially daunting in the case of meshed systems because of the possibly huge number of parameters. As a consequence, network issues are usually disregarded, either completely (single-bus models) or partially (by assuming perfect competition for the transmission services or by assuming that agents do not anticipate the effect of their actions on transmission pricing). On the other hand, network congestions could have a great impact on market outcomes, as relatively isolated load pockets supplied by a reduced number of generators are subject to a greater threat from the exercise of market power. Therefore, ignoring the impact of congestions on conjectural parameters values that intend to quantify market power could result in an unacceptable approximation [13].

The aim of this paper is to advance in the understanding of a particular set of conjectural parameters, namely the conjectural price response, in relationship to the congestion status of the grid. The main result is that conjectural price responses fulfill, under some assumptions related to the market organization, a symmetry property: the conjectural price response of bus i with respect to injections in bus j has the same value as that of bus j with respect to injections in bus i. Moreover, the matrix formed by arranging conjectural price responses is a positive-definite one. The work reported in this paper generalizes the one in [14].

The rest of the paper is organized as follows. The next section describes assumptions underlying market organization. Section III introduces the concepts of residual demand and residual welfare. Then, in section IV, the definition and computation of conjectural price responses are addressed. Sections V and VI include the main results: conjectural response matrix is symmetric and positive-definite. Finally, I conclude.

II. Market description

A. General market organization

The market is assumed to be organized as initially suggested in [15]. That is, generators tender electricity from each one of their generating facilities, and consumers submit bids in each network node. Payments are made in accordance with nodal prices. An auctioneer clears the market trying, to maximize apparent social welfare, to be precisely defined below. Therefore, the market is based in implicit transmission auctions. Congestion rents are appropriated by the ISO, the TSO or other non-strategic bodies. A single period is assumed, as inter-temporal constraints are assumed not to be taken into account by the market mechanism.

More explicitly, the market is described as having

- n_g oligopolistic generators (utilities).

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- n_u generating units. P_{gu} is the power output of unit u belonging to generator g. This unit is placed in bus $i(u)$ (each generator may have several units in several buses, and in each bus there may be units belonging to several generators). The unit is characterized by a maximum output $P_{gu}^{\max}$. The minimum output is 0. Each unit submits a tender $P_{gu} = O_{gu}(\pi_{i(u)})$, $\pi_{i(u)}$ being the electricity price at node $i(u)$ and O_{gu} an increasing function.
- A network made by n_{bus} buses and n_{lin} transmission lines. Each line is characterized by an admittance y_l and a maximum flow $f_l^{\max}$, so that the actual flow f_l fulfills $-f_l^{\max} \leq f_l \leq f_l^{\max}$. DC power flow equations are assumed, so that if line l goes from bus $i(l)$ to bus $j(l)$, the flow fulfills $f_l = y_l\left(\theta_{i(l)} - \theta_{j(l)}\right)$. θ_i is the voltage phase at bus i. No losses are considered.
- A demand at each bus i which fulfills D_i. The consumers submit a bid $D_i = B_i(\pi_i)$, which is a decreasing function.

Generators submit tenders and consumers bid simultaneously. They are cleared by a central auctioneer by maximizing apparent social welfare subject to network constraints, as shown below.

B. The apparent social welfare

From generating plants' tenders, the apparent cost function can be defined as:

$$C_{gu}(P_{gu}) = \int_0^{P_{gu}} O_{gu}^{-1}(P_{gu})\, dP_{gu} \tag{1}$$

As $O_{gu}(\pi_{i(u)})$ is increasing, $C_{gu}(P_{gu})$ is convex. Note that this function is the real cost function only if the price tendered is equal to the marginal cost, which is not usually the case when analyzing oligopolistic competition. The apparent cost of a generator is the sum of the apparent costs of the units belonging to it:

$$C_g = \sum_{u \in \mathcal{U}_g} C_{gu} \tag{2}$$

$\mathcal{U}_g$ being the set of units that belong to generator g. Analogously, demand utility can be computed from demand bids as

$$U_i(D_i) = \int_0^{D_i} B_i^{-1}(D_i)\, dD_i \tag{3}$$

As $B_i(\pi_i)$ is decreasing, $C_{gu}(P_{gu})$ is concave. Moreover, total demand utility is defined as

$$U = \sum_i U_i \tag{4}$$

The apparent social welfare is the difference between total utility and total apparent cost:

$$W = U - \sum_g C_g \tag{5}$$

C. Market clearing conditions

Market clearing is done by the auctioneer in order to maximize apparent social welfare. In order to write the mathematical conditions in a easily manipulable form, let us introduce the following notation:

- $\mathbf{P}_g$ and $\mathbf{O}_g$ are vectors containing outputs and tenders for all the units belonging to generator g:

$$\mathbf{P}_g = \begin{bmatrix} P_{gu_1} \\ P_{gu_2} \\ \vdots \end{bmatrix}, \quad \mathbf{O}_g = \begin{bmatrix} O_{gu_1} \\ O_{gu_2} \\ \vdots \end{bmatrix}$$

- $\mathbf{D}$ and $\mathbf{B}$ are vectors containing the demands and customers bids at each bus.
- $\mathcal{C}_g$ is a 0-1 matrix which maps units into buses. That is, if element (i,j) of matrix $\mathcal{C}_g$ is 1, that means that unit gu_j is placed at bus i.
- $\boldsymbol{\pi}$ is the vector containing the nodal prices:

$$\boldsymbol{\pi} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \end{bmatrix}$$

- $\boldsymbol{\theta}$ is the vector containing the phases at the different buses, except bus 1 whose phase is set to 0.

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_2 \\ \theta_3 \\ \vdots \end{bmatrix}$$

- $\mathbf{f}$ is a vector containing the flows. $\mathbf{f}^{\max}$ contains their limits.
- $\mathcal{F}$ is the matrix, obtained from the admittance data, relating flows and phases: $\mathbf{f} = \mathcal{F}\boldsymbol{\theta}$.
- $\mathcal{M}$ is the bus-line incidence matrix ($\mathcal{M}(i,j) = 1$ if line j is leaving bus i, and -1 if it is arriving at bus i).

The auctioneer problem can be formulated as

$$\begin{array}{lc} \max_{\mathbf{P}_g,\mathbf{D},\mathbf{f},\boldsymbol{\theta}} & U(\mathbf{D}) - \sum_g C_g(\mathbf{P}_g) \\ \text{s.t.} & \left\{ \begin{array}{rcl} \mathbf{D} + \mathcal{M}\mathbf{f} & = & \sum_g \mathcal{C}_g \mathbf{P}_g \ : \ \boldsymbol{\pi} \\ \mathbf{f} & = & \mathcal{F}\boldsymbol{\theta} \\ -\mathbf{f}^{\max} \leq & \mathbf{f} & \leq \mathbf{f}^{\max} \end{array} \right. \end{array} \tag{6}$$

The problem is well defined, as it amounts to maximizing a concave function on a convex set, defined by the DC-flow network constraints. Nodal prices $\boldsymbol{\pi}$ are the multipliers of the load-generation balance constraints $\mathbf{D} + \mathcal{M}\mathbf{f} = \sum_g \mathcal{C}_g \mathbf{P}_g$. First order optimality conditions imply that

$$\nabla_{\mathbf{P}_g} C_g(\mathbf{P}_g) = \mathcal{C}_g^T \boldsymbol{\pi} \tag{7}$$

$$\nabla_{\mathbf{D}} U(\mathbf{D}) = \boldsymbol{\pi} \tag{8}$$

As, from (1) and (3), $\nabla_{\mathbf{P}_g} C_g(\mathbf{P}_g) = \mathbf{O}_g^{-1}(\mathbf{P}_g)$ and $\nabla_{\mathbf{D}} U(\mathbf{D}) = \mathbf{B}^{-1}(\mathbf{D})$,

$$\mathbf{O}_g^{-1}(\mathbf{P}_g) = \mathcal{C}_g^T \boldsymbol{\pi} \tag{9}$$

$$\mathbf{B}^{-1}(\mathbf{D}) = \boldsymbol{\pi} \tag{10}$$

So, the auctioneer clears the market as it fulfills network constraints.

III. The residual demand

At this point it is useful to introduce the "residual demand" of generator g in bus i. It is defined from the demand bid and generator g competitors' tenders:

$$R_{gi}(\pi_i) = B_i(\pi_i) - \sum_{u \in \mathcal{U}_{g'}/g' \neq g, i(u)=i} O_{g'u}(\pi_{i(u)}) \tag{11}$$

Residual social welfare is defined as

$$W_g(\mathbf{Q}_g) = \sum_i \int_0^{Q_{gi}} R_{gi}^{-1}(Q_{gi})\, dQ_{gi} \tag{12}$$

Variables Q_{gi} merit some discussion. On the hand, they have been introduced as the "variables" related to the inverse residual demand functions R_{gi}. Dimensional analysis shows that they have the same dimensions as $\mathbf{D}$ or $\mathbf{P}_g$, that is, MW-h. On the other hand, they are also the demand at bus i not satisfied by generators other than g. This follows from residual demand definition as excess demand in equation (11). Consequently, demand balance equation can also be written:

$$\mathbf{Q} + \mathcal{M}\mathbf{f} = \mathcal{C}_g \mathbf{P}_g \tag{13}$$

Reduced welfare W_g is a concave function, as reduced demands R_{gi} are decreasing functions. Moreover, from the point of view of the auctioneer, the aggregate effect of consumers and generators other than g is summarized in the residual demand function. Therefore, market clearing conditions can be written as the solution of the optimization problem:

$$\begin{array}{ll} \max_{\mathbf{P}_g, \mathbf{Q}_g, \mathbf{f}, \boldsymbol{\theta}} & W_g(\mathbf{Q}_g) - C_g(\mathbf{P}_g) \\ \text{s.t.} & \left\{ \begin{array}{rcl} \mathbf{Q}_g + \mathcal{M}\mathbf{f} & = & \mathcal{C}_g \mathbf{P}_g \ : \ \boldsymbol{\pi} \\ \mathbf{f} & = & \mathcal{F}\boldsymbol{\theta} \\ -\mathbf{f}^{\max} \leq & \mathbf{f} & \leq \mathbf{f}^{\max} \end{array} \right. \end{array} \tag{14}$$

First order optimality conditions imply that

$$\nabla_{\mathbf{P}_g} C_g(\mathbf{P}_g) = \mathcal{C}_g^T \boldsymbol{\pi} \tag{15}$$

$$\nabla_{\mathbf{Q}_g} W_g(\mathbf{Q}_g) = \boldsymbol{\pi} \tag{16}$$

In the remainder of this paper, it will be assumed that demand bids B_i and generators' offers O_i are strictly monotonic and smooth functions. Apparently this one is too stringent a condition, as real bids and offers are usually step-wise functions. However, because of market uncertainties, conjectural bids and tenders are more likely to be modeled as smooth functions. For instance, most Conjectural Supply Function approaches make use of affine functions. In any case, the assumptions above imply that residual welfare is a strictly concave function and that its second derivative is well defined. Therefore, around the clearing market point $\mathbf{Q}_g^*$, and up to second order:

$$W_g(\mathbf{Q}_g^* + \delta\mathbf{Q}_g) = W_g(\mathbf{Q}_g^*) + \boldsymbol{\pi}^T \delta\mathbf{Q}_g - \frac{1}{2}\delta\mathbf{Q}_g^T \mathcal{A}_g \delta\mathbf{Q}_g \tag{17}$$

matrix positive-definite: $\mathcal{A}_g > 0$ being symmetric.

IV. Computing price responses

Generator g conjectural price response are the anticipated sensitivities of prices with respect to power injections, assuming specific demand and competitors behaviour. The responses can be organized in matrix form $\mathcal{S}_g$, the element (i, j) being the sensitivity of price at node i with respect to power injection in node j:

$$\mathcal{S}_g(i,j) = -\frac{\partial \pi_i}{\partial P_j} \Big|_{\delta\mathbf{P}_g = \mathbf{0}} \tag{18}$$

The minus sign is included because usually a price decrease is anticipated whenever power is injected in the grid. The qualification $\delta\mathbf{P}_g = \mathbf{0}$ means that generator g keeps its production constant. As a consequence of additional P_j injection, prices are going to change, because the other generators as well as the consumers are going to modify their injections or withdrawals in accordance with their tenders and bids. In most models, the relevant tenders and bids are possibly not the real ones, but rather those assumed (or "conjectured") by generator g. Hence the term "conjectural price responses".

The relevance of price responses come from their role in specifying the marginal revenue of generator g. Equilibrium models require marginal revenues and costs to be equal. Marginal costs can be computed from the different companies cost functions. Although the task can be challenging, marginal cost computation is a problem which engineers have been addressing for a long time. On the other hand, marginal revenue is the sum of two terms. The first one is the market price, as this is the price paid to an additional MW of generated output. The second term is the effect that the additional MW has on the market price paid to the remaining generated output (the infra-marginal one). Oligopoly simulation most serious difficulties usually arise from this second term specification. Price responses are a parameter that can be helpful in this regard.

Problem (17) is the starting point in order to compute the sought sensitivities. Perturbations around the market clearing point have to be considered. Furthermore, variables $\mathbf{P}_g$ are to be treated as constants, as generator g is assumed to keep production constant. Finally, injections $\delta\mathbf{P}$ in the different nodes have to be considered. Therefore, we are led to study the problem

$$\begin{array}{ll} \max_{\mathbf{Q}_g,\mathbf{f},\boldsymbol{\theta}} & W_g(\mathbf{Q}_g) \\ \text{s.t.} & \left\{ \begin{array}{rcl} \mathbf{Q}_g + \mathcal{M}\mathbf{f} & = & \mathcal{C}_g\mathbf{P}_g + \delta\mathbf{P} \; : \; \boldsymbol{\pi} \\ \mathbf{f} & = & \mathcal{F}\boldsymbol{\theta} \\ -\mathbf{f}^{\max} \leq & \mathbf{f} & \leq \mathbf{f}^{\max} \end{array} \right. \end{array} \tag{19}$$

Let us introduce incremental variables around the market clearing solution $(\mathbf{Q}_g^*, \mathbf{f}^*, \boldsymbol{\theta}^*)$ defined by

$$\mathbf{Q}_g = \mathbf{Q}_g^* + \delta\mathbf{Q}_g \tag{20}$$

$$\mathbf{f} = \mathbf{f}^* + \delta\mathbf{f} \tag{21}$$

$$\boldsymbol{\theta} = \boldsymbol{\theta}^* + \delta\boldsymbol{\theta} \tag{22}$$

Let us also consider constrained lines, that is, lines that in the market clearing point have power-flows at their limits ($f_l = f_l^{\max}$ or $f_l = f_l^{\min}$). It is assumed that they are congested lines and that they remain congested in the event of a small incremental injection. This is not a harmless assumption, as the oligopolistic game outcome can result in lines being "critically congested". In these lines the flow is at its maximum, but arbitrarily small changes in the system conditions can result in line being either congested or not congested [16, 17]. Analysis of these cases can be carried out along the lines suggested in this paper, although is considerably more cumbersome. In any case, under the assumption of non-critically congested lines, it can be written that for any congested line $\delta f_l = 0$. Let us introduce a 0-1 matrix $\mathcal{E}$, with the number of rows equal to the number of congested lines, and the number of columns equal to the number of lines, such that the conditions above can be stated as

$$\mathcal{E}\delta\mathbf{f} = \mathbf{0} \tag{23}$$

After taking into account (17), problem (19) can be finally re-stated as

$$\begin{array}{ll} \max_{\delta\mathbf{Q}_g,\delta\mathbf{f},\delta\boldsymbol{\theta}} & \boldsymbol{\pi}^T\delta\mathbf{Q}_g - \frac{1}{2}\delta\mathbf{Q}_g^T\mathcal{A}_g\delta\mathbf{Q}_g \\ \text{s.t.} & \left\{ \begin{array}{rcl} \delta\mathbf{Q}_g + \mathcal{M}\delta\mathbf{f} & = & \delta\mathbf{P} \\ \delta\mathbf{f} & = & \mathcal{F}\delta\boldsymbol{\theta} \\ \mathcal{E}\delta\mathbf{f} & = & \mathbf{0} \end{array} \right. \end{array} \tag{24}$$

By introducing multipliers $\boldsymbol{\lambda}_\mathbf{D}$, $\boldsymbol{\lambda}_\mathbf{f}$ and $\boldsymbol{\lambda}_\mathbf{s}$, the first order optimality conditions can be written:

$$\begin{bmatrix} \mathcal{A}_g & 0 & 0 & 0 & 0 & \mathcal{I} \\ 0 & 0 & 0 & \mathcal{F}^T & 0 & 0 \\ 0 & 0 & 0 & -\mathcal{I} & \mathcal{E}^T & \mathcal{M}^T \\ 0 & \mathcal{F} & -\mathcal{I} & 0 & 0 & 0 \\ 0 & 0 & \mathcal{E} & 0 & 0 & 0 \\ \mathcal{I} & 0 & \mathcal{M} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\mathbf{Q}_g \\ \delta\boldsymbol{\theta} \\ \delta\mathbf{f} \\ \boldsymbol{\lambda}_\mathbf{f} \\ \boldsymbol{\lambda}_\mathbf{s} \\ \boldsymbol{\lambda}_\mathbf{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\pi} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \delta\mathbf{P} \end{bmatrix} \tag{25}$$

where $\mathcal{I}$ is an identity matrix of the required dimension.

In order to compute $\mathcal{S}_g$, it is useful to compute the matrix $\mathcal{S}_{Qg}$, whose element (i,j) is the change in the residual demand at bus i given a marginal power injection at bus j. As

$$\frac{\partial \pi_i}{\partial Q_{gi}} = R_{gi}^{-1} \tag{26}$$

it follows that

$$\mathcal{S}_g = \mathcal{A}_g\mathcal{S}_{Qg} \tag{27}$$

On the other hand, matrix $\mathcal{S}_{Qg}$ can be computed by solving

$$\begin{bmatrix} \mathcal{A}_g & 0 & 0 & 0 & 0 & \mathcal{I} \\ 0 & 0 & 0 & \mathcal{F}^T & 0 & 0 \\ 0 & 0 & 0 & -\mathcal{I} & \mathcal{E}^T & \mathcal{M}^T \\ 0 & \mathcal{F} & -\mathcal{I} & 0 & 0 & 0 \\ 0 & 0 & \mathcal{E} & 0 & 0 & 0 \\ \mathcal{I} & 0 & \mathcal{M} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{S}_{Qg} \\ \vdots \end{bmatrix} = \begin{bmatrix} \boldsymbol{\pi}\mathbf{1}^T \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathcal{I} \end{bmatrix} \tag{28}$$

where $\mathbf{1}$ is a column vector, all of whose elements are 1.

V. PRICE RESPONSE MATRIX IS SYMMETRIC

Let us define

$$\mathcal{B} = [0, 0, 0, 0, \mathcal{I}] \tag{29}$$

and

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & \mathcal{F}^T & 0 & 0 \\ 0 & 0 & -\mathcal{I} & \mathcal{E}^T & \mathcal{M}^T \\ \mathcal{F} & -\mathcal{I} & 0 & 0 & 0 \\ 0 & \mathcal{E} & 0 & 0 & 0 \\ 0 & \mathcal{M} & 0 & 0 & 0 \end{bmatrix} = \mathcal{C}^T \tag{30}$$

Note that $\mathcal{C}$ is symmetric. So, (25) is written as

$$\begin{bmatrix} \mathcal{A}_g & \mathcal{B} \\ \mathcal{B}^T & \mathcal{C} \end{bmatrix} \begin{bmatrix} \delta \mathbf{Q}_g \\ \vdots \end{bmatrix} = \begin{bmatrix} \boldsymbol{\pi} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \delta \mathbf{P} \end{bmatrix} \tag{31}$$

$\delta \mathbf{Q}_g$ should be a linear function of $\boldsymbol{\pi}$ and $\delta \mathbf{P}$. However, it only depends on $\delta \mathbf{P}$. To prove this, let us consider (31) with $\delta \mathbf{P} = \mathbf{0}$. Then $\delta \mathbf{Q}_g = \mathbf{0}$, as no redistribution of residual demand can improve the reduced welfare. On the other hand, (31) solution is the superposition of the two solutions with $\delta \mathbf{P} = \mathbf{0}$ and $\boldsymbol{\pi} = \mathbf{0}$ respectively. First solution has $\delta \mathbf{Q}_g = \mathbf{0}$, for the reason given above. T Therefore, if only $\delta \mathbf{Q}_g$ is to be computed, the specific value of $\boldsymbol{\pi}$ is irrelevant, and it is possible to set $\boldsymbol{\pi} = \mathbf{0}$. Then, after formal manipulation of (31),

$$\left(\mathcal{A}_g - \mathcal{B}\mathcal{C}^{-1}\mathcal{B}^T\right) \delta \mathbf{Q}_g = -\mathcal{B}\mathcal{C}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \delta \mathbf{P} \end{bmatrix} \tag{32}$$

The problem here is that $\mathcal{C}$ is a singular matrix. On the other hand, it is always possible to build a sequence $\mathcal{C}(\epsilon)$ of symmetric regular matrices whose limit as $\epsilon \to 0$ is $\mathcal{C}$. So, let us define $\mathcal{C}(\epsilon)$ as

$$\mathcal{C}(\epsilon) = \epsilon \mathcal{I} + \mathcal{C} \tag{33}$$

This perturbation moves the whole $\mathcal{C}$ spectrum by ϵ. As the spectrum is a discrete set, it is clear that there is a ϵ_1 such that for any ϵ in the interval $(0, \epsilon_1)$, matrix $\mathcal{C}(\epsilon)$ is not singular. We will restrict ourselves in the following to this interval.

So, let us define $\delta \mathbf{Q}_g(\epsilon)$ as the solution of

$$\left(\mathcal{A}_g - \mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\right) \delta \mathbf{Q}_g(\epsilon) = -\mathcal{B}\mathcal{C}(\epsilon)^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \delta \mathbf{P} \end{bmatrix} \tag{34}$$

Therefore the matrix $\mathcal{S}_{Qg}(\epsilon)$ (the sensitivity matrix relating increment $\delta \mathbf{Q}_{gi}(\epsilon)$ to power injection at bus j) fulfills

$$\begin{aligned} \left(\mathcal{A}_g - \mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\right) \mathcal{S}_{Qg}(\epsilon) &= -\mathcal{B}\mathcal{C}(\epsilon)^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathcal{I} \end{bmatrix} \\ &= -\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T \end{aligned} \tag{35}$$

So,

$$\begin{aligned} \mathcal{S}_{Qg}(\epsilon) &= -\left(\mathcal{A}_g - \mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\right)^{-1} \mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T \\ &= -\left(\mathcal{I} - \mathcal{A}_g^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\right)^{-1} \mathcal{A}_g^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T \\ &= -\left[\mathcal{I} + \mathcal{A}_g^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T + \right. \\ &\qquad \left. \left(\mathcal{A}_g^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\right)^2 + \ldots\right] \mathcal{A}_g^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T \\ &= -\mathcal{A}_g^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T - \\ &\qquad \mathcal{A}_g^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\mathcal{A}_g^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T - \\ &\qquad \mathcal{A}_g^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\mathcal{A}_g^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\mathcal{A}_g^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T + \ldots \end{aligned}$$

And as a consequence

$$\begin{aligned} \mathcal{S}_g(\epsilon) &= \mathcal{A}_g\mathcal{S}_{Qg}(\epsilon) \\ &= -\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T - \mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\mathcal{A}_g^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T - \\ &\qquad \mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\mathcal{A}_g^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\mathcal{A}_g^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T + \ldots \end{aligned}$$

which is a clearly symmetric matrix. Taking the limit $\epsilon \to 0$, it is shown that $\mathcal{S}_g$ is symmetric.

VI. PRICE RESPONSE MATRIX IS POSITIVE-DEFINITE

Let us partition $\mathcal{C}(\epsilon)$ as

$$\mathcal{C}(\epsilon) = \begin{bmatrix} \epsilon & 0 & \mathcal{F}^T & 0 & 0 \\ 0 & \epsilon & -\mathcal{I} & \mathcal{E}^T & \mathcal{M}^T \\ \mathcal{F} & -\mathcal{I} & \epsilon & 0 & 0 \\ 0 & \mathcal{E} & 0 & \epsilon & 0 \\ 0 & \mathcal{M} & 0 & 0 & \epsilon \end{bmatrix} = \begin{bmatrix} \epsilon\mathcal{I} & \mathcal{C}_r^T \\ \mathcal{C}_r & \epsilon\mathcal{I} \end{bmatrix} \tag{36}$$

where

$$\mathcal{C}_r = \begin{bmatrix} \mathcal{F} & -\mathcal{I} \\ 0 & \mathcal{E} \\ 0 & \mathcal{M} \end{bmatrix} \tag{37}$$

Now, it is easy to check

$$\mathcal{C}(\epsilon)^{-1} = \begin{bmatrix} \epsilon\mathcal{I} & -\mathcal{C}_r^T \\ -\mathcal{C}_r & \epsilon\mathcal{I} \end{bmatrix} \begin{bmatrix} \left(\epsilon^2\mathcal{I} - \mathcal{C}_r^T\mathcal{C}_r\right)^{-1} & 0 \\ 0 & \left(\epsilon^2\mathcal{I} - \mathcal{C}_r\mathcal{C}_r^T\right)^{-1} \end{bmatrix} \tag{38}$$

On the other hand, let us write matrix $\mathcal{B}$ as

$$\mathcal{B} = [0,0,0,0,\mathcal{I}] = [0,\mathcal{J}] \tag{39}$$

with

$$\mathcal{J} = [0,\mathcal{I}] \tag{40}$$

So,

$$\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T = \epsilon\mathcal{J}\left(\epsilon^2\mathcal{I} - \mathcal{C}_r^T\mathcal{C}_r\right)^{-1}\mathcal{J}^T \tag{41}$$

It is easy to check that

$$\mathcal{C}_r^T\mathcal{C}_r = \left[\begin{array}{cc} \mathcal{F}^T\mathcal{F} & -\mathcal{F}^T \\ -\mathcal{F} & \mathcal{I} + \mathcal{E}^T\mathcal{E} + \mathcal{M}^T\mathcal{M} \end{array}\right] \tag{42}$$

$\mathcal{C}_r^T\mathcal{C}_r$ is a strictly positive-definite matrix. To check this, let us consider

$$\begin{aligned} \left[\mathbf{v}_{\boldsymbol{\theta}}^T, \mathbf{v}_{\mathbf{f}}^T\right]\mathcal{C}_r^T\mathcal{C}_r\left[\begin{array}{c}\mathbf{v}_{\boldsymbol{\theta}} \\ \mathbf{v}_{\mathbf{f}}\end{array}\right] &= \mathbf{v}_{\boldsymbol{\theta}}^T\mathcal{F}^T\mathcal{F}\mathbf{v}_{\boldsymbol{\theta}} + \\ &\quad \mathbf{v}_{\mathbf{f}}^T\left(\mathcal{I} + \mathcal{E}^T\mathcal{E} + \mathcal{M}^T\mathcal{M}\right)\mathbf{v}_{\mathbf{f}} - \\ &\quad 2\mathbf{v}_{\boldsymbol{\theta}}^T\mathcal{F}^T\mathbf{v}_{\mathbf{f}} \\ &= \left(\mathbf{v}_{\boldsymbol{\theta}}^T\mathcal{F}^T - \mathbf{v}_{\mathbf{f}}^T\right)\left(\mathcal{F}\mathbf{v}_{\boldsymbol{\theta}} - \mathbf{v}_{\mathbf{f}}\right) + \\ &\quad \mathbf{v}_{\mathbf{f}}^T\left(\mathcal{E}^T\mathcal{E} + \mathcal{M}^T\mathcal{M}\right)\mathbf{v}_{\mathbf{f}} \end{aligned}$$

It is obvious that this expression is greater than or equal to zero. If we assume that it is zero, then $\mathcal{M}\mathbf{v}_{\mathbf{f}} = \mathbf{0}$ and $\mathcal{F}\mathbf{v}_{\boldsymbol{\theta}} = \mathbf{v}_{\mathbf{f}}$. These equations describe a DC electricity network with flows $\mathbf{v}_{\mathbf{f}}$, phases $\mathbf{v}_{\boldsymbol{\theta}}$ and null injections. Therefore, all the flows must be zero and all the phases must be equal. As $\theta_1 = 0$, all the phases must be zero. So, we have proved that if the expression above is zero, then $\mathbf{v}_{\mathbf{f}} = \mathbf{0}$ and $\mathbf{v}_{\boldsymbol{\theta}} = \mathbf{0}$, which means that $\mathcal{C}_r^T\mathcal{C}_r > 0$ as claimed.

Let $\epsilon_2 > 0$ be the smallest $\mathcal{C}_r^T\mathcal{C}_r$ eigenvalue. Then, if $\epsilon < \min(\sqrt{0.5\epsilon_2}, \epsilon_1)$, matrix $\left(\epsilon^2\mathcal{I} - \mathcal{C}_r^T\mathcal{C}_r\right)$ must be definite-negative, and therefore matrix $\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T$ in equation (41) (which is the restriction to a lower dimensional subspace) must be also definite-negative.

Let us define $\mathcal{C}_{\mathcal{B}}(\epsilon) = \mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T$. $\mathcal{C}_{\mathcal{B}}(\epsilon)$ spectrum is bounded. If the $\mathcal{C}_r^T\mathcal{C}_r$ spectrum is in the interval $[\epsilon_2, E_2]$, $\mathcal{C}_{\mathcal{B}}(\epsilon)$ must be in $[-E_2, -0.5\epsilon_2] = [-K, -k]$, because $0.5\epsilon_2$ is the greatest perturbation that $\epsilon^2\mathcal{I}$ can induce in $-\mathcal{C}_r^T\mathcal{C}_r$. Note that $K, k > 0$ are two constants independent of ϵ.

Now, $-K\mathcal{I} < \mathcal{C}_{\mathcal{B}}(\epsilon) < -k\mathcal{I}$, so $-k^{-1}\mathcal{I} < \mathcal{C}_{\mathcal{B}}(\epsilon)^{-1} < K^{-1}\mathcal{I}$. Let us denote by $\underline{\alpha}$ and $\overline{\alpha}$ the smallest and greatest values of $\mathcal{A}_g$. They are positive real numbers, as $\mathcal{A}_g$ is a symmetric definite-positive matrix. So $\underline{\alpha}\mathcal{I} < \mathcal{A}^{-1} < \overline{\alpha}\mathcal{I}$. Then

$$\left(\underline{\alpha} + K^{-1}\right)\mathcal{I} < \mathcal{A}_g^{-1} - \mathcal{C}_{\mathcal{B}}(\epsilon)^{-1} < \left(\overline{\alpha} + k^{-1}\right)\mathcal{I}$$

And, therefore

$$\left(\mathcal{A}_g^{-1} - \mathcal{C}_{\mathcal{B}}(\epsilon)^{-1}\right)^{-1} > \left(\overline{\alpha} + k^{-1}\right)^{-1}\mathcal{I} = \bar{k}\mathcal{I} > 0$$

But,

$$\begin{aligned} \left(\mathcal{A}_g^{-1} - \mathcal{C}_{\mathcal{B}}(\epsilon)^{-1}\right)^{-1} &= \mathcal{C}_{\mathcal{B}}(\epsilon)\left(\mathcal{A}_g^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon) - \mathcal{I}\right)^{-1} \\ &= -\mathcal{C}_{\mathcal{B}}(\epsilon)\left(\mathcal{I} - \mathcal{A}_g^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon)\right)^{-1} \\ &= -\mathcal{C}_{\mathcal{B}}(\epsilon)\left(\mathcal{I} + \mathcal{A}_g^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon) + \right. \\ &\quad \left. \mathcal{A}_g^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon)\mathcal{A}_g^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon) + \ldots\right) \\ &= -\mathcal{C}_{\mathcal{B}}(\epsilon) - \mathcal{C}_{\mathcal{B}}(\epsilon)\mathcal{A}_g^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon) - \\ &\quad \mathcal{C}_{\mathcal{B}}(\epsilon)\mathcal{A}_g^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon)\mathcal{A}_g^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon) - \ldots \\ &= \mathcal{S}(\epsilon) \end{aligned}$$

Therefore, $\mathcal{S}(\epsilon) > \bar{k}\mathcal{I}$, and as a consequence $\mathcal{S} > 0$.

VII. Conclusions

It has been shown that the conjectural price response matrix must be a symmetric and positive-definite one. This fact is relevant as it makes possible to put constraints on the possible values that these parameters can take and to design effective algorithms for market equilibrium computing.

VIII. References

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IX. BIOGRAPHY

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