Equivalent Reduced Order Systems trough Generalized Selective Modal Analysis

Julián Barquín
Member, IEEE

Abstract—**This paper presents some new theoretical developments in order to obtain reduced order equivalent systems of very big linear dynamic systems, such as those that model small-signal phenonema in power systems. The proposed approach preserves input-output relationships, which allows its use for designing control systems. The methodology is based on Selective Modal Analysis framework, and more specifically in its generalized form.**

Keywords: **Power system stability, Control systems design.**

I. Introduction

Selective Modal Analysis (SMA) is a general theoretical framework for the analysis of large linear dynamic systems [1]. It is particularly useful in systems, such as the power systems, where there is an intuition on the physical nature of the relevant modes [2]. For instance, electromechanical modes can be related to oscillations of a certain set of generators rotors against other set [3].

Control design techniques are considerably more efficient when dealing with systems of relatively low dimension, and understanding and critique of results much deeper and sharper. On the other hand, power systems models are very large ones, including typically thousands of variables. One early aim of Selective Modal Analysis has been to obtain reduced order systems for the interesting dynamics, which allows a more efficient control design [4, 5].

A recent development in SMA has been the so-called Generalized SMA [6, 7]. Generalized SMA allows a more efficient use of the physical information above quoted, obtained either from the engineer intuition or from other simplified models (for instance, the classical model in relation to the whole system model, including controllers).

The purpose of this paper is to show how Generalized SMA techniques can be applied to obtain reduced order systems for control design. The focus of the paper is a mainly theoretical one.

J. Barquín is with Instituto de Investigación Tecnológica, Universidad Pontificia Comillas, C/ Alberto Aguilera 23, 28015 Madrid, Spain. e-mail: barquin@iit.upco.es.

The sequel is as follows. In section II basic model reduction approach and formulae are introduced. Next, remarks concerning the proper way of incorporating known physical behaviour in the reduction approach (namely, the subspaces choosing procedure) are made. Section IV deals with the efficient computation of the reduced model. Then, an algorithm is proposed in section V, and demonstrated in a test system in section VI. Finally, I conclude.

II. System reduction

Our starting point will be a dynamical linear system, with one input and one output variable.

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{1}$$
$$y = \mathbf{c}^T\mathbf{x} + du \tag{2}$$

It is not difficult to generalize the results for multi-input and multi-output systems, although I will stick to the single input single output case to avoid cluttering the notation.

In the equation above, $\mathbf{x}, \mathbf{b}, \mathbf{c} \in \Re^m$, $\mathbf{A}, \mathbf{E} \in \Re^{m\times m}$. Besides, $\mathbf{E}$ is a symmetric idempotent matrix:

$$\mathbf{E} = \mathbf{E}^T = \mathbf{E}^2 \tag{3}$$

Usually, $\mathbf{E}$ has the special structure:

$$\mathbf{E} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{4}$$

being $\mathbf{I}$ an identity matrix of suitable dimensions. In this case, the first subset of components of $\mathbf{x}$ are *bona fide* state variables, and the last ones *static* or *algebraic* variables, which time derivative does not actually appears in the dynamic equations.

In "classical" SMA a subset of variables is chosen as relevant ones, and a system reduction procedure is organized around this selection. In Generalized SMA, two subspaces must be chosen. The way of defining these subspaces is by providing two basis that span both subspaces:

$$\mathcal{E} = [\mathbf{e}_1, \ldots, \mathbf{e}_n] \quad (5)$$
$$\mathcal{F} = [\mathbf{f}_1, \ldots, \mathbf{f}_n] \quad (6)$$

So, $\mathcal{E}, \mathcal{F} \in \Re^{m\times n}$. The order of the reduced system is going to be n, so that it will be imposed that $n \ll m$. It will be proven in the sequel $\mathcal{E}$ and $\mathcal{F}$ are related to, respectively, controllability and observability system aspects. Moreover, $\mathcal{E}, \mathcal{F}$ are normalized such that

$$\mathcal{F}^T \mathbf{E} \mathcal{E} = \mathbf{I} \quad (7)$$

After Laplace transforming (1)-(2) and some algebraic transformations, it is obtained that:

$$s\mathbf{a} = (\mathbf{A}_r + \mathbf{H}(s))\, \mathbf{a} + \tilde{\mathbf{L}}(s)^T \mathbf{b} u \quad (8)$$
$$y = \mathbf{c}^T \tilde{\mathbf{K}}(s)\mathbf{a} + \left(d + \mathbf{c}^T \mathbf{N}(s)\mathbf{b}\right) u \quad (9)$$

where $\mathbf{a} \in \Re^n$ is a reduced state vector, and the reduced matrices are defined as:

$$\mathbf{A}_r = \mathcal{F}^T \mathbf{A} \mathcal{E} \quad (10)$$
$$\mathbf{H}(s) = \mathcal{F}^T \mathbf{A} \mathbf{P} \left\{ s\mathbf{E} - \mathbf{A} + \mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A} \right\}^{-1} \mathbf{P} \mathbf{A} \mathcal{E} \quad (11)$$
$$\tilde{\mathbf{L}}(s)^T = \mathcal{F}^T + \mathcal{F}^T \mathbf{A} \mathbf{P} \left\{ s\mathbf{E} - \mathbf{A} + \mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A} \right\}^{-1} \mathbf{P} \quad (12)$$
$$\tilde{\mathbf{K}}(s) = \mathcal{E} + \mathbf{P} \left\{ s\mathbf{E} - \mathbf{A} + \mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A} \right\}^{-1} \mathbf{P} \mathbf{A} \mathcal{E} \quad (13)$$
$$\mathbf{N}(s) = \mathbf{P} \left\{ s\mathbf{E} - \mathbf{A} + \mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A} \right\}^{-1} \mathbf{P} \quad (14)$$

where projection matrices $\mathbf{P}$ and $\mathbf{Q}$ are defined as:

$$\mathbf{P} = \mathbf{I} - \mathbf{E}\mathcal{E}\mathcal{F}^T\mathbf{E} = \mathbf{I} - \mathbf{Q} \quad (15)$$

III. Choosing the relevant subspaces

Formulae above are most useful if matrices $\mathbf{H}(s)$, $\tilde{\mathbf{L}}(s)^T\mathbf{b}$, $\mathbf{c}^T\tilde{\mathbf{K}}(s)$, $\mathbf{c}^T\mathbf{N}(s)\mathbf{b}$ can be chosen to be independent of s. It is possible to approach this objective by an appropriate choice of subspaces spanned by $\mathcal{E}$ and $\mathcal{F}$.

If $\mathbf{E}\mathbf{b} \neq \mathbf{0}$, a vector $\mathbf{v}_0$ can be defined as $\mathbf{v}_0 = \mathbf{E}\mathbf{b}$ and matrix $\mathcal{E}$ chosen in such as way that $\mathbf{v}_0 \in \text{span}(\mathcal{E})$. In this case, $\mathbf{P}\mathbf{b} = \mathbf{0}$, and therefore $\mathbf{c}^T\mathbf{N}(s)\mathbf{b} = 0$ and $\tilde{\mathbf{L}}(s)^T\mathbf{b} = \mathcal{F}^T\mathbf{b}$. An analogous procedure in case that $\mathbf{c}^T\mathbf{E} \neq \mathbf{0}^T$ (to define $\mathbf{w}_0 = \mathbf{E}^T\mathbf{c}$ and thereafter impose $\mathbf{w}_0 \in \text{span}(\mathcal{F})$) can be used to make $\mathbf{c}^T\mathbf{N}(s)\mathbf{b} = 0$ and $\mathbf{c}^T\tilde{\mathbf{K}}(s) = \mathbf{c}^T\mathcal{E}$.

In the most general case, relevant subspaces can be chosen in the following way. Matrix $\mathbf{E}$ can be written as:

$$\mathbf{E} = \mathcal{E}^* \mathcal{F}^{*T} \quad (16)$$

where matrices $\mathcal{E}^*$ and $\mathcal{F}^*$ are analogous (they also provide subspaces basis) to matrices $\mathcal{E}$ and $\mathcal{F}$ above. For instance, if $\mathbf{E}$ is like in (4):

$$\mathcal{E}^* = \mathcal{F}^* = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} \quad (17)$$

Let be $\tilde{\mathbf{L}}(s)^*$ and $\tilde{\mathbf{K}}(s)^*$ the matrices computed by using $\mathcal{E}^*$ and $\mathcal{F}^*$ instead of $\mathcal{E}$ and $\mathcal{F}$ in (12)-(13). Let us define vectors $\mathbf{v}$ and $\mathbf{w}$ as

$$\mathbf{v}_0 = \mathcal{E}^* \tilde{\mathbf{L}}(s)^{*T} \mathbf{b} \quad (18)$$
$$\mathbf{w}_0^T = \mathbf{c}^T \tilde{\mathbf{K}}(s)^* \mathcal{F}^{*T} \quad (19)$$

Then, by imposing $\mathbf{v}_0 \in \text{span}(\mathcal{E})$ and $\mathbf{w}_0 \in \text{span}(\mathcal{F})$, matrices $\tilde{\mathbf{L}}(s)^T\mathbf{b}$, $\mathbf{c}^T\tilde{\mathbf{K}}(s)$ and $\mathbf{c}^T\mathbf{N}(s)\mathbf{b}$ are rendered independent of s.

So, it only remains making $\mathbf{H}(s)$ independent of s. Actually, this is impossible, as $\mathbf{A}_r + \mathbf{H}(s)$ must contain the whole system dynamics, that is

$$\det\left(s\mathbf{I} - \mathbf{A}_r - \mathbf{H}(s)\right) = 0 \quad (20)$$

must be fulfilled whenever s is one of the many eigenvalues of pair $(\mathbf{E}, \mathbf{A})$. On the other hand, it is not expected that most of these modes can be excited by input u or observed from output y, bust just the ones which controllability and observability factors are high enough.

For these modes, adequacy of matrices $\mathcal{E}$ and $\mathcal{F}$ is measured by the participation ratio. Specifically, participation ratio ρ_i of mode $i = 1, \ldots, r$ is given by

$$\rho_i = -\frac{\mathbf{w}_i^T \mathbf{Q} \mathbf{v}_i}{\mathbf{w}_i^T \left(\mathbf{E} - \mathbf{Q}\right) \mathbf{v}_i} \quad (21)$$

where $\mathbf{v}_i$ and $\mathbf{w}_i$ are the right and left eigenvectors of the mode i. Participation factor ρ_i is infinite if either $\mathbf{v}_i$ belongs to the span of $\mathcal{E}$ or $\mathbf{w}_i$ to that of $\mathcal{F}$. In that case, $\mathbf{A}_r + \mathbf{H}(s)$ exactly represents the i-th mode dynamics.

In order to design a control system, relevant modes can be identified by small-signal analysis in the system without control, or possibly in a simplified model of it as the classical model. In that way, a set of (possibly approximate) relevant eigenvectors $\mathbf{v}_i, \mathbf{w}_i, i = 1, \ldots, r$ is identified.

Once this task has been performed, matrices $\mathcal{E}$ and $\mathcal{F}$ are chosen by imposing that vectors $\mathbf{v}_j, \mathbf{w}_j, i = 0, \ldots, r$ belong to their span. So, matrices $\tilde{\mathbf{L}}(s)^T\mathbf{b}$, $\mathbf{c}^T\tilde{\mathbf{K}}(s)$, $\mathbf{c}^T\mathbf{N}(s)\mathbf{b}$ are independent of s and matrix $\mathbf{H}(s)$ is almost constant for the relevant dynamic, and can be numerically evaluated for a suitable s_0. The resulting reduced system can be now used in order to design the control.

IV. Matrix computations

Formulae (11)-(14), although quite interesting from a theoretical point of view, are not generally suitable to perform actual computations. Main reason is that matrix $s\mathbf{E} - \mathbf{A} + \mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A}$ is not sparse, even if $\mathbf{E}$ and $\mathbf{A}$ are.

On the other hand, there are alternative expressions which only involve the inverse of $\mathbf{E}s - \mathbf{A}$. As these matrices are ultimately multiplied by vectors, to have a factorization of $\mathbf{E}s - \mathbf{A}$ is enough for computational purposes.

In order to derive these expressions, let us introduce matrix $\mathbf{M}(s)$:

$$\mathbf{M}(s) = \mathcal{F}^T \mathbf{E} \left(\mathbf{A} - s\mathbf{E}\right)^{-1} \mathbf{E}\mathcal{E} \tag{22}$$

Note than $\mathbf{M}(s)$ is a small dimension matrix, so it is not difficult to factorize. Then:

$$\begin{aligned}
\mathbf{H}(s) &= \mathbf{M}(s)^{-1} + s\mathbf{I} - \mathbf{A}_r & (23)\\
\tilde{\mathbf{K}}(s) &= \left(\mathbf{A} - s\mathbf{E}\right)^{-1} \mathbf{E}\mathcal{E}\mathbf{M}(s)^{-1} & (24)\\
\tilde{\mathbf{L}}(s)^T &= \mathbf{M}(s)^{-1}\mathcal{F}^T \mathbf{E} \left(\mathbf{A} - s\mathbf{E}\right)^{-1} & (25)\\
\mathbf{N}(s) &= \tilde{\mathbf{K}}(s)\mathbf{M}(s)\tilde{\mathbf{L}}(s)^T - \left(\mathbf{A} - s\mathbf{E}\right)^{-1} & (26)
\end{aligned}$$

Unfortunately, these expressions do not generally apply to solve (18)-(19). The reason is that, in this case, matrices $\mathcal{E}^*$ and $\mathcal{F}^*$ have a great number of columns, so associated matrix $\mathbf{M}(s)^*$ is too big to be easily factorized. Two approaches are possible:

1. To use (24)-(25) as starting point to compute $\mathbf{v}_0$ and $\mathbf{w}_0$ by an iterative procedure that does not require $\mathbf{M}(s)^*$ factorization.
2. To take advantage of any special structure of $\mathbf{E}$, like the one in (4).

In the example shown in this paper, $\mathbf{Eb} \neq \mathbf{0}$ and $\mathbf{c}^T\mathbf{E} \neq \mathbf{0}^T$, so that results in the previous section can be applied in order to show that $\tilde{\mathbf{K}}(s)$, $\tilde{\mathbf{L}}(s)$ and $\mathbf{K}(s)$ are independent of s. A detailed study of both alternatives will be the subject of a future paper.

V. Reduction algorithm

The former theoretical results can be applied in order to propose a reduction algorithm. In this paper it will be assumed that $\mathbf{Eb} \neq \mathbf{0}$ and $\mathbf{c}^T\mathbf{E} \neq \mathbf{0}^T$. So, only matrix $\mathbf{H}(s)$ actually depends of s.

Following classical SMA procedures, it will be required that $\mathbf{H}(s)$ exactly contains the interesting dynamics. That is, given a set $\lambda_i, i = 1, \ldots, l$ of interesting eigenvalues of the original full dynamical system, the associated set of reduced eigenvectors $\mathbf{a}_i$ are defined so that they fulfil:

$$\lambda_i \mathbf{a}_i = \left(\mathbf{A}_r + \mathbf{H}(\lambda_i)\right) \mathbf{a}_i \tag{27}$$

It is easy to show that reduced eigenvectors are related to full eigenvectors $\mathbf{v}_i$ by means of equation:

$$\mathcal{F}^T \mathbf{E}\mathbf{v}_i = \mathbf{a}_i \tag{28}$$

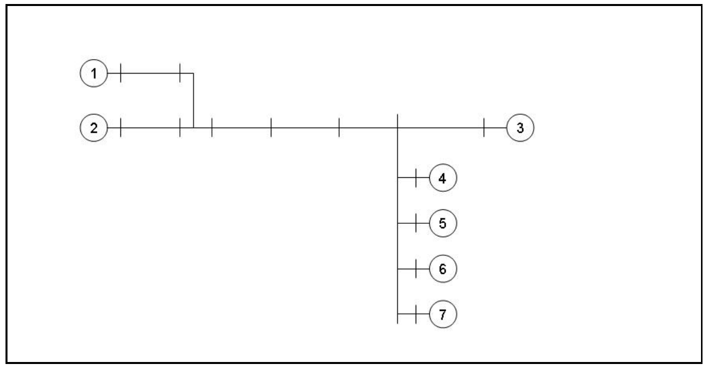

Fig. 1. Example case network

The reduced system state matrix is required to fulfil:

$$\mathbf{A}_{\text{red}}\mathbf{a}_i = \left(\mathbf{A}_r + \mathbf{H}(\lambda_i)\right) \mathbf{a}_i, \quad i = 1, \ldots, l \tag{29}$$

This equation is a single real equation in case that λ_i is real, but two real equations if λ_i is complex (its real and imaginary parts). So, the number of interesting real eigenvalues plus twice the number of complex ones must be not greater than the reduced order matrix dimension n. In the algorithm here proposed they are equal, so that matrix $\mathbf{A}_{\text{red}}$ is essentially unique.

Basic algorithm idea is to initially assume two reduced subspaces $\mathcal{E}$ and $\mathcal{F}$ where the interesting dynamic is presumed to lie, and some estimation of the value of the interesting eigenvalues λ_i^0. With this data, matrix $\mathbf{A}_r + \mathbf{H}(\lambda_i^0)$ is computed, associated reduced eigenvectors $\mathbf{a}_i^0$ identified, and a reduced matrix $\mathbf{A}_{\text{red}}^0$ proposed. Eigenanalysis of this matrix provides a new estimation of the interesting eigenvalues λ_i^1, and the whole procedure is iterated until convergence.

A more formal algorithm description is shown in the pseudo code shown in the appendix. Based in the theoretical results in [7], it can be proven that the algorithm converge at least at a a linear rate.

VI. Example case

In order to check the proposed algorithms, a small system has been analysed. Its one-line diagram is displayed at figure 1. There are seven generators, and the total number of state variables is 91. There are also 42 algebraic variables. Therefore, dimension $m = 133$.

The study focuses in an electromechanical mode characterized by oscillations of generator 3 versus the coherent group of generators 4 to 6. In order to design a controller it is assumed that the observed variable (y) is the velocity ω_3 of generator 3, and the input u is applied to the mechanical equation $\dot{\omega}_3 = \ldots$ via, for instance, the field current modulation. These two conditions fix vectors $\mathbf{b}$ and $\mathbf{c}$.

Subspace matrices $\mathcal{E}$ and $\mathcal{F}$ include, respectively, vectors $\mathbf{b}$ and $\mathbf{c}$, that in this case happen to be equal. In order to complete the basis, the following vectors are added to $\mathcal{E}$:

- $\mathbf{e}_2$ is zero, but the component associated to the generator 3 angle, that is set to 1.
- $\mathbf{e}_3$ is zero, but the components associated to the generators 4, 5 and 6 angles, that are set to 1.
- $\mathbf{e}_4$ is like $\mathbf{e}_3$, but with the velocities components instead of the angles.
- $\mathbf{e}_5$ is zero, but the component associated to the field current of generator 3.

In this example, $\mathcal{F} = \mathcal{E}$, so vectors $\mathbf{f}_i$ are associated to the dynamic equations $\dot{x} = \ldots$, being x the state variable associated to $\mathbf{e}_i$. Vectors $\mathbf{e}_i$ and $\mathbf{f}_i$ are further normalized in order that $\mathcal{F}^T \mathbf{E} \mathcal{E} = \mathbf{I}$.

To initialize the reduction algorithm, the interesting modes are set to $\lambda^0 = \{\pm j3, \pm j6, -1\}$. The reduction algorithm converge in 5 iterations to the system shown at the bottom of the page.

The Bode diagram of both the full and the reduced system are shown in figure 2. Note that both diagrams are almost equal in the figure frequency range, which allows to design a control system for the full system by using the reduced system.

VII. Conclusions

In this paper a system reduction technique, based on Generalized Selective Modal Analysis, has been shown. The technique exploits relationships between physical variables and relevant modes for a given control schema. New theoretical results pertaining to the SMA reduction approach have been shown. A reduction algorithm has been proposed and demonstrated for a test system.

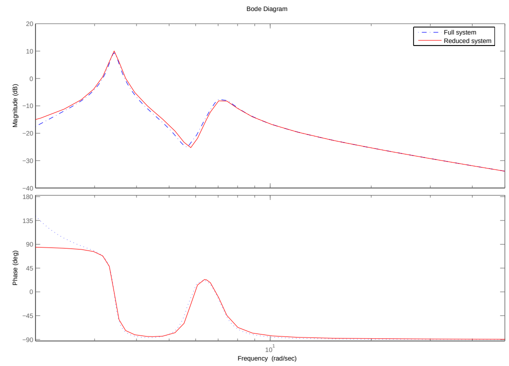

Fig. 2. Bode diagram. Solid line: reduced system, dotted line: full system

VIII. Acknowledgements

The help given by Dr. Luis Rouco, by providing the test system data, as well as valuable remarks, is gratefully acknowledged.

IX. References

[1] I. J. Pérez-Arriaga, *Selective Modal Analysis With Applications to Electric Power Systems*, Ph. D. Thesis, Electrical Engineering, M. I. T., June, 1981.

[2] I.J. Pérez-Arriaga, G.C. Verghese, F.C. Schweppe, "Selective Modal Analysis with Applications to Electric Power Systems. Part I: Heuristic Introduction. Part II: The Dynamic Stability Problem". *IEEE Transactions on Power Apparatus and Systems*, Vol. PAS-101, No. 9, September 1982, pp. 3117-3134.

[3] P. Kundur, Power System Stability and Control. Mc Graw Hill, New York 1994.

[4] F.L. Pagola, I.J. Pérez-Arriaga, G.C. Verghese, "On Sensitivities, Residues and Participations". *IEEE Transactions on Power Systems*, Vol. PWRS-4, No. 1, February 1989, pp. 278-285.

$$\begin{bmatrix} \dot{a}_1 \\ \dot{a}_2 \\ \dot{a}_3 \\ \dot{a}_4 \\ \dot{a}_5 \end{bmatrix} = \begin{bmatrix} -0.4458 & -0.0715 & 0.0280 & 0.4107 & 0.0058 \\ 377.0105 & 0.0004 & 0.0002 & 0.0093 & 0.0001 \\ 0.0336 & 0.0007 & 0.0003 & 377.0072 & 0.0002 \\ 0.8397 & 0.1000 & -0.0863 & -0.5484 & 0.0106 \\ -268.8430 & -0.5370 & -0.4068 & -136.2081 & -2.9081 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [1, 0, 0, 0, 0] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

[5] L. Rouco, F.L. Pagola, "An Eigenvalue Sensitivity Approach to Location and Controller Design Controllable Series Capacitors for Damping Power Systems Electromechanical Oscillations". IEEE Transactions on Power Systems, Vol. PWRS-12, No. 4 November 1997, pp. 1660-1666.

[6] J. Barquin, L. Rouco, H. R. Vargas, "Generalized selective modal analysis", 2002 IEEE/PES Winter Meeting: Panel Session on Recent Applications of Linear Analysis

[7] J. Barquín. "Generalized Selective Modal Analysis". Available at `http://xxx.lanl.gov`, paper *math.LA/9906043*, July, 1999.

I. Algorithm pseudocode

Reduction Algorithm

Input: $\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}, d, \mathcal{E}, \mathcal{F}, \{\lambda_i^0\}$.

Output: $\mathbf{A}_{\text{red}}, \mathbf{b}_{\text{red}}, \mathbf{c}_{\text{red}}, d_{\text{red}}$.

Requirements: $\mathbf{Eb} \neq \mathbf{0}, \mathbf{c}^T\mathbf{E} \neq \mathbf{0}^T$.
  Number of real interestinng eigenvalues plus
  twice the number of complex ones
  must be the common dimension
  of subspaces spanned by $\mathcal{E}$ and $\mathcal{F}$

  Form $\mathbf{A}_r = \mathcal{F}^T\mathbf{A}\mathcal{E}$.
  for $j = 0, 1, 2, \ldots$ until convergence,
    Set $k = 1$.
    for $i = 1, 2, \ldots$
      Compute $\mathbf{H}(s_i^j)$ by using equation (23).
      Perform the $\mathbf{A}_r + \mathbf{H}(s_i^j)$ eigenalysis.
      Choose as s_i^{j+1} the computed eigenvalue
        closer to s_i^j.
      Choose as $\mathbf{a}_i^{j+1}$ the associated eigenvector.
      if s_i^{j+1} **is real**
        Define $\mathbf{r}_k = \left(\mathbf{A}_r + \mathbf{H}(s_i^j)\right)\mathbf{a}_i^{j+1}$
        Define $\mathbf{l}_k = \mathbf{a}_i^{j+1}$
        Set $k = k + 1$
      else if s_i^{j+1} **is complex**
        Define $\mathbf{r}^C = \left(\mathbf{A}_r + \mathbf{H}(s_i^j)\right)\mathbf{a}_i^{j+1}$
        Define $\mathbf{l}^C = \mathbf{a}_i^{j+1}$
        Set $\mathbf{r}_k = \Re\{\mathbf{r}^C\}$
        Set $\mathbf{r}_{k+1} = \Im\{\mathbf{r}^C\}$
        Set $\mathbf{l}_k = \Re\{\mathbf{l}^C\}$
        Set $\mathbf{l}_{k+1} = \Im\{\mathbf{l}^C\}$
        Set $k = k + 2$
      end
    end
    Solve $\mathbf{A}_{\text{red}}^{j+1}[\mathbf{l}_1, \mathbf{l}_2, \ldots] = [\mathbf{r}_1, \mathbf{r}_2, \ldots]$
    Check $\mathbf{A}_{\text{red}}$ convergence condition.
  end
  Compute $\mathbf{b}_{\text{red}} = \mathcal{F}^T\mathbf{b}$
  Compute $\mathbf{c}_{\text{red}}^T = \mathbf{c}^T\mathcal{E}$
  Compute $d_{\text{red}} = d$