9th International Conference on Probabilistic Methods Applied to Power Systems
KTH, Stockholm, Sweden – June 11-15, 2006

Critically-congested Nash-Cournot equilibria in electricity networks

Julián Barquín

Abstract—**Oligopolistic generators behaviour may result in equilibria characterized by critically-congested lines, that is, congested lines such that arbitrarily small generation changes cause them to be not-congested. The nature of such equilibria, under the Cournot (quantity bids) hypothesis is discussed, and an algorithm proposed to compute them. Probabilities measures arise in a natural way in the analysis and in the proposed algorithm.**

Keywords: **Power system economics, Interconnected power systems.**

I. Introduction

Real electricity markets are not textbook perfect competition markets. They rather exhibit, in lesser or greater degree, oligopolistic characteristics that may jeopardize their social role as means for an efficient generation and use of electricity. Therefore, there is a growing need of tools aimed to quantify their shortcomings and the likely impact of possible regulatory measures or of the lack of them.

Cournot, or Nash-Cournot equilibria, stems from idealized electricity markets in which the different generators decide a certain amount of power to be generated. Demand is assumed to be price-responsive, so its level is adjusted in order to maximize its utility. In this way, prices and, therefore, generator profits are established. Generators anticipate the demand and theirs competitors behaviour in order to decide their generation level. This level is assumed to be the Nash equilibrium of the described game.

Nash-Cournot equilibria do not seem to provide realistic forecasts of the electricity market. However, they are useful to asses the opportunities for market power exercise of different agents, and to suggest possible regulatory measures. They are also a first step for more realistic models (as those based on conjectured prices responses). It can even be argued that, by modelling demand in a proper way, they can provide realistic long-term forecasts of electricity markets.

Not surprisingly, a number of models have been developed around the world to compute them [1, 2]. This task is far from being a trivial one, specially if a realistic representation of the generation equipment is required, possibly including realistic cost functions, or hydro and other intetemporal constraints.

The existence of significant transmission constraints adds another layer of complexity. It is also worrisome from the point of view of possible market power exercise. For instance, when one or more lines of the system are congested, isolated areas may appear, and consequently, some demand can only be supplied by a reduced subset of producers. Then, there will be increased market power, since it is relatively easy for these producers to increase prices in the isolated area by withdrawing power, independently of the competitiveness of the whole system.

It is usual, in this kind of studies, to model the transmission network by using DC loadflow equations. In this setting, two kinds of Nash-Cournot equilibria can be found. In type I equilibria, small variations of generation levels do not change any line status, i.e., if it is congested or not. Instead, type II equilibria are located in a point where one or more lines are critically congested, that is, arbitrarily small changes in the generators outputs can change its status from being not congested to congested or vice versa. This solution can arise in a natural (generic) way, because the outcome of the generators game can lead them to "play around the congestion".

This paper can be read as a continuation of [3] (the described model therein is also more briefly described in [2]), in which a model for the computation of the first kind of equilibria is proposed. Now, the model is enhanced to deal as well with the second kind. This is done by introducing a set of probabilities, that may be interpreted as a formal device to compute these equilibria, although they also can have a "real" probabilistic sense.

The sequel of the paper is organized as follows. Next section shows the used notation and reviews some previous results that are needed in the sequel. Section III introduces basic ideas of the new algorithm, and section IV the actual implementation. Next, a small test system is studied. Finally, I conclude.

II. Type I equilibrium computation

This section summarizes the main results in [3] that will be required in the sequel.

Julián Barquín is with Instituto de Investigación Tecnológica, Universidad Pontificia Comillas, C/ Alberto Aguilera 25, 28015 Madrid, Spain. e-mail: julian.barquin@iit.upcomillas.es.

A. Main assumptions and notation

The assumed model is that of a single period, purely thermal system. It is further considered

- n_g generators (utilities) that behave as Cournot oligopolists, i.e., set their units outputs (MW) in order to maximize their profit assuming known demand curves and fixed competitors' outputs.
- n_u thermal units. P_{gu} is the power output of unit u belonging to generator g. This unit is placed in bus $i(u)$ (each generator may have several units in several buses, and in each bus may be units belonging to several generators). The unit is characterized by a maximum output $P_{gu}^{\max}$, and a convex cost function $C_{gu} = C_{gu}(P_{gu})$. The minimum output is 0.
- A network made by n_{bus} buses and n_{lin} transmission lines. Each line is characterized by an admittance y_l and a maximum flow $f_l^{\max}$, so that the actual flow f_l fulfils $-f_l^{\max} \leq f_l \leq f_l^{\max}$. DC power flow equations are assumed, so that if line l goes from bus $i(l)$ to bus $j(l)$, the flow fulfils $f_l = y_l \left(\theta_{i(l)} - \theta_{j(l)}\right)$. θ_i is the voltage phase at bus i. No losses are considered
- A demand at each bus i which fulfils $D_i = D_{0i} - \alpha_i \pi_i$, being D_i the actual demand (MW), π_i the price (€/MW-h), and D_{0i} and α_i known constants.

In addition, the following symbols will be used:

- $\mathbf{P}_g$ is a column vector containing all the units belonging to generator g: $\mathbf{P}_g = [P_{gu_1}, P_{gu_2}, \ldots]^T$.
- C_g is the total operating cost of generator g: $C_g = \sum_{gu} C_{gu}$.
- $\mathcal{C}_g$ is a 0-1 matrix which maps units into buses. That is, if element (i,j) of matrix $\mathcal{C}_g$ is 1, that means that unit gu_j is placed at bus i.
- $\boldsymbol{\pi}$ is the column vector containing the nodal prices: $\boldsymbol{\pi} = [\pi_1, \pi_2, \ldots]^T$.
- $\mathcal{S}$ is a matrix, whose element (i,j) is the minus change of the price in bus j when an additional MW is injected in bus i, assuming that generated outputs are hold constant (only demands are allowed to change): $\mathcal{S}(i,j) = -\frac{\partial \pi_i}{\partial P_j}$. The demand variations are such that total demand utility (over all the buses) is maximized.
- $\mathcal{S}_D$ is similar to $\mathcal{S}$, but when demand sensitivities instead of price sensitivities are sought: $\mathcal{S}_D(i,j) = \frac{\partial D_i}{\partial P_j}$.
- $\mathcal{A}$ is a diagonal matrix, whose element $\mathcal{A}(i,i) = \frac{1}{\alpha_i}$.
- $\mathcal{E}$ is a 0-1 matrix, with number of rows equal to the number of constrained lines and columns equal to the number of lines. Element $\mathcal{E}(l,j) = 0$ if the l-th congested line is the line number j.
- $\mathbf{D}$ is a vector containing the demands at each bus.
- U_i is the bus i demand dis-utility: $U_i = \frac{1}{2\alpha_i}\left(D_{0i} - D_i\right)^2$.
- U is the total demand dis-utility $U = \sum_i U_i$.
- $\boldsymbol{\theta}$ is the vector containing the phases at the different buses, but the bus 1 whose phase is set to 0: $\boldsymbol{\theta} = [\theta_2, \theta_3, \ldots]^T$.
- $\mathbf{f}$ a vector containing the flows. $\mathbf{f}^{\max}$ contains the limits.
- $\mathcal{F}$ is the matrix, obtained from the admittance data, relating flows and phases: $\mathbf{f} = \mathcal{F}\boldsymbol{\theta}$.
- $\mathcal{M}$ is the buses-lines incidence matrix ($\mathcal{M}(i,j) = 1$ if line j is leaving bus i, and -1 if it is arriving at bus i).

The generators simultaneously submit bids for each one of their units. Each bid is a quantity bid, the amount of power to be produced. A central auctioneer receives the bids and decides the amount of demand to be consumed by each bus. This is done by maximizing the consumers utility subject to fulfil the network constraints.

B. Generators' behaviour

Each generator sets the output of is units in order to maximize its profit. Therefore, in equilibrium, marginal revenue and cost must be equal:

$$MR_{gu} = MC_{gu} \tag{1}$$

Marginal cost is just the derivative $\frac{dC_{gu}}{dP_{gu}}$. Marginal revenue has two components: the price $\pi_{i(gu)}$ at the bus where unit gu is located, and the decrease of the profit in each bus where generator u has units because of the decrease in the price because of the additional generation $\frac{d\pi_{i(u')}}{dP_{gu}} P_{gu'}$. Therefore, assuming that the unit is operating somewhere between its limits, $\frac{dC_{gu}}{dP_{gu}} = \pi_{i(u)} + \sum_{gu'} \frac{d\pi_{i(u')}}{dP_{gu}} P_{gu'}$. This expression can be written in vectorial form for each generator g:

$$\nabla_{\mathbf{P}_g} C_g(\mathbf{P}_g) + \mathcal{C}_g^T \mathcal{S} \mathcal{C}_g \mathbf{P}_g = \mathcal{C}_g^T \boldsymbol{\pi} \tag{2}$$

where $\mathcal{C}_g^T$ is the transpose of $\mathcal{C}_g$.

C. Optimization approach

In equilibrium, the demand equations as well as the network equations must be also fulfilled. The demand equations are $D_i = D_{0i} - \alpha_i \pi_i$, that can be written as: $\nabla_{\mathbf{D}} U(\mathbf{D}) = -\boldsymbol{\pi}$. Adding equation (2) and network equations, the following equilibrium equationas are obtained:

$$\begin{array}{rcl} \nabla_{\mathbf{P}_g} C_g(\mathbf{P}_g) + \mathcal{C}_g^T \mathcal{S} \mathcal{C}_g \mathbf{P}_g & = & \mathcal{C}_g^T \boldsymbol{\pi} \\ \nabla_{\mathbf{D}} U(\mathbf{D}) & = & -\boldsymbol{\pi} \\ \mathbf{D} + \mathcal{M}\mathbf{f} & = & \sum_g \mathcal{C}_g \mathbf{P}_g \\ \mathbf{f} & = & \mathcal{F}\boldsymbol{\theta} \\ -\mathbf{f}^{\max} \leq & \mathbf{f} & \leq \mathbf{f}^{\max} \end{array}$$

In [3] it is proven that matrix $\mathcal{S}$ is positive symmetric. *Assuming that it is given*, it is immediate to check that the above equations are the optimality conditions of the problem[1]:

$$\begin{array}{ll} \min_{\mathbf{P}_g, \mathbf{D}, \mathbf{f}, \boldsymbol{\theta}} & \sum_g C_g(\mathbf{P}_g) + \frac{1}{2} \mathbf{P}_g^T \mathcal{C}_g^T \mathcal{S} \mathcal{C}_g \mathbf{P}_g + U(\mathbf{D}) \\ \text{s.t.} & \left\{ \begin{array}{rcl} \mathbf{D} + \mathcal{M}\mathbf{f} & = & \sum_g \mathcal{C}_g \mathbf{P}_g \\ \mathbf{f} & = & \mathcal{F}\boldsymbol{\theta} \\ \mathbf{0} \leq & \mathbf{P}_g & \leq \mathbf{P}_g^{\max} \\ -\mathbf{f}^{\max} \leq & \mathbf{f} & \leq \mathbf{f}^{\max} \end{array} \right. \end{array} \quad (3)$$

The problem is a well defined one, as C_{gu} and U are convex functions and $\mathcal{S}$ a symmetric positive matrix (see appendices). Prices $\boldsymbol{\pi}$ are the multipliers of demand-generation constraints $\mathbf{D} + \mathcal{M}\mathbf{f} = \sum_g \mathcal{C}_g \mathbf{P}_g$.

D. $\mathcal{S}$ computation

In problem (3) it is assumed that $\mathcal{S}$ is given. But $\mathcal{S}$ depends on the system flows (specifically, on which lines are constrained). Therefore, an issue of consistency arises. The purpose of this subsection is to show how, from the nodal prices and knowledge of the constrained lines, matrix $\mathcal{S}$ can be computed.

$\mathcal{S}$ is the price sensitivity to a marginal increase in the injected power. Therefore, incremental changes $\delta\mathbf{D}$, $\delta\boldsymbol{\pi}$, $\delta\boldsymbol{\theta}$ and so on will be considered.

Basic idea is that, if any generator changes its bid in an amount $\delta\mathbf{P}$, after clearing the market the new operating point will be a minimum of the dis-utility function. As $\boldsymbol{\pi} = -\nabla_{\mathbf{D}} U(\mathbf{D})$, then $U(\mathbf{D} + \delta\mathbf{D}) = U(\mathbf{D}) + \boldsymbol{\pi}^T \delta\mathbf{D} + \frac{1}{2} \delta\mathbf{D}^T \mathcal{A} \delta\mathbf{D}$. So, the demand increments are found by solving

[1] At this point the bounds on generator outputs have been reintroduced. It is also easy to check (albeit a bit cumbersome) that in that way the first order optimality conditions are the same equations than the equilibrium ones when limits are considered

$$\begin{array}{ll} \min_{\delta\mathbf{D}} & \boldsymbol{\pi}^T \delta\mathbf{D} + \frac{1}{2} \delta\mathbf{D}^T \mathcal{A} \delta\mathbf{D} \\ \text{s.t.} & \left\{ \begin{array}{rcl} \delta\mathbf{D} + \mathcal{M}\delta\mathbf{f} & = & \delta\mathbf{P} \\ \delta\mathbf{f} & = & \mathcal{F}\boldsymbol{\delta\theta} \\ \mathcal{E}\delta\mathbf{f} & = & \mathbf{0} \end{array} \right. \end{array}$$

The last constraint forces the changes in the congested lines flows to be zero. It is clear that this constraint only makes sense for kind I equilibria. Being a quadratic problem, its optimality conditions are a linear system. By making the vector $\delta\mathbf{P}$ to have all components zero, but the one corresponding in the single bus j, demand sensitivities to injections in bus j are computed. More formally, matrix $\mathcal{S}_D$ is computed by solving

$$\begin{bmatrix} \mathcal{A} & 0 & 0 & 0 & 0 & \mathcal{I} \\ 0 & 0 & 0 & \mathcal{F}^T & 0 & 0 \\ 0 & 0 & 0 & -\mathcal{I} & \mathcal{E}^T & \mathcal{M}^T \\ 0 & \mathcal{F} & -\mathcal{I} & 0 & 0 & 0 \\ 0 & 0 & \mathcal{E} & 0 & 0 & 0 \\ \mathcal{I} & 0 & \mathcal{M} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{S}_D \\ \vdots \end{bmatrix} = \begin{bmatrix} -\boldsymbol{\pi}\mathbf{1}^T \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathcal{I} \end{bmatrix} \quad (4)$$

where $\mathbf{1}$ is a column vector, all which elements are 1. Finally, matrix $\mathcal{S}$ is computed as:

$$\mathcal{S} = \mathcal{A}\mathcal{S}_D \quad (5)$$

E. The algorithm for type I equilibria

The algorithm proposed in [3] amounts essentially to iterate between equations (3), and (5) and (4). So, an initial constraints' state for each line is assumed, and equations (5) and (4) are solved to compute $\mathcal{S}$. Then, problem (3) is solved, and the lines' status (constrained on or off) assessed. If this state is the same one than the formerly assumed, the algorithm ends. If not, the new lines' status computed when solving (3) is plugged in (5) and (4), and the whole procedure iterated.

III. The enhanced algorithm: general remarks

Specification of network state (required for matrix $\mathcal{S}$ specification) amounts to the specification of every line status (congested or not congested). In that sense, the previous algorithm can be seen as a map from the discrete set of network states onto itself. Therefore, only two outcomes are possible: either the algorithm converges to a single state in a finite number of iterations, or it converges to a limit cycle.

In the first case, the algorithm output is a type I Nash-Cournot equilibrium. The second case is the starting point of the algorithm described in this section, aimed to compute type II equilibria.

In a type II equilibrium one or more lines are critically congested, that is, its flow is maximum (or minimum) and its associated multipliers are zero. In this case the matrix $\mathcal{S}$ is not well defined, as a small change resulting in an uncongested line has prices sensitivities different of the ones of a change that congests even more the line (by making the multiplier strictly positive). So, basic equilibrium equation (2) does not make sense any longer.

These difficulties can be overcome if a probabilistic setting is considered, even if it is in a purely formal way. Because of unavoidable uncertainties in the market, there are circumstances when generators cannot be certain on the status of some line (congested or not congested). We can expect that, in this case, generators try to maximize their expected profit, which implies that expected cost and income are to be equal:

$$\mathbb{E}_g\left[MR_{gu}\right] = \mathbb{E}_g\left[MC_{gu}\right] \tag{6}$$

where $\mathbb{E}_g$ denotes the expectation for generator g. Note that each generator can use a different probability distribution (either because they have access to different information, or because it is desired to model different risk aversion characteristics, or for any other reason). However, it is required for the different probability distributions to be equivalent, i.e., they assign probability zero to the same set (they deem as impossible the same outcomes).

Previously, we derived equilibrium equation (2) from marginal cost and revenue equality (1). Analogously, from (6) it is possible to conclude that in equilibrium:

$$\nabla_{\mathbf{P}_g} C_g(\mathbf{P}_g) + \mathcal{C}_g^T \mathbb{E}_g\left[\mathcal{S}\right] \mathcal{C}_g \mathbf{P}_g = \mathcal{C}_g^T \mathbb{E}_g\left[\boldsymbol{\pi}\right] \tag{7}$$

as generated outputs $\mathbf{P}_g$ are decision variables known to each agent.

I will focus in the limit of vanishingly uncertainty. In this limit, the price uncertainty is going to be also vanishingly small, as prices are continuous functions of generated outputs. However, sensitivity matrix $\mathcal{S}$ is discontinuous in the point where a line becomes congested. So, in the limit, random variable $\boldsymbol{\pi}$ becomes non-random, whereas matrix $\mathcal{S}$ remains random. So, equation (7) becomes $\nabla_{\mathbf{P}_g} C_g(\mathbf{P}_g) + \mathcal{C}_g^T \mathbb{E}_g\left[\mathcal{S}\right] \mathcal{C}_g \mathbf{P}_g = \mathcal{C}_g^T \boldsymbol{\pi}$. As previously, demand and network equations must be added, to obtain the equilibrium equations:

$$\begin{aligned}
\nabla_{\mathbf{P}_g} C_g(\mathbf{P}_g) + \mathcal{C}_g^T \mathbb{E}_g\left[\mathcal{S}\right] \mathcal{C}_g \mathbf{P}_g &= \mathcal{C}_g^T \boldsymbol{\pi} \\
\nabla_{\mathbf{D}} U(\mathbf{D}) &= -\boldsymbol{\pi} \\
\mathbf{D} + \mathcal{M}\mathbf{f} &= \sum_g \mathcal{C}_g \mathbf{P}_g \\
\mathbf{f} &= \mathcal{F}\boldsymbol{\theta} \\
-\mathbf{f}^{\max} \leq \quad \mathbf{f} &\leq \mathbf{f}^{\max}
\end{aligned}$$

Assuming matrices $\mathbb{E}_g\left[\mathcal{S}\right]$ as given, previous system are the optimality conditions of problem[2]:

$$\begin{array}{ll}
\min_{\mathbf{P}_g, \mathbf{D}, \mathbf{f}, \boldsymbol{\theta}} & \sum_g C_g(\mathbf{P}_g) + \frac{1}{2}\mathbf{P}_g^T \mathcal{C}_g^T \mathbb{E}_g\left[\mathcal{S}\right] \mathcal{C}_g \mathbf{P}_g + U(\mathbf{D}) \\
\text{s.t.} & \left\{ \begin{array}{rcl}
\mathbf{D} + \mathcal{M}\mathbf{f} &=& \sum_g \mathcal{C}_g \mathbf{P}_g : \boldsymbol{\pi} \\
\mathbf{f} &=& \mathcal{F}\boldsymbol{\theta} \\
\mathbf{0} \leq & \mathbf{P}_g & \leq \mathbf{P}_g^{\max} \\
-\mathbf{f}^{\max} \leq & \mathbf{f} & \leq \mathbf{f}^{\max} : \boldsymbol{\mu}_{\min}, \boldsymbol{\mu}_{\max}
\end{array} \right.
\end{array} \tag{8}$$

As previously, prices $\boldsymbol{\pi}$ are the multipliers of the demand constraint. I have also introduced the multipliers of the maximum flow constraints $\boldsymbol{\mu}_{\min}, \boldsymbol{\mu}_{\max}$. These inequality multipliers are non-negative.

Given a set of state probabilities ps_s^g, matrix $\mathbb{E}_g\left[\mathcal{S}\right]$ can be written as:

$$\mathbb{E}_g\left[\mathcal{S}\right] = \sum_s ps_s^g \mathcal{S}_s$$

As every $\mathcal{S}_s$ is positive, $\mathbb{E}_g\left[\mathcal{S}\right]$ is positive as well. Therefore, objective function in (8) is convex, and the optimization problem is well defined.

Equivalent probabilities for each agent are required. So, a set of weights w_s^g and a "natural" probability measure ps_s are specified such that

$$ps_s^g = \frac{1}{\sum_s w_s^g ps_s} w_s^g ps_s$$

In previous section algorithm, consistency between the computed solution of (3) and assumed matrix $\mathcal{S}$ was required. A similar consistency condition is required now. To establish this condition, the relationship between the probability of any line l to be congested (pl_l) must be related to the set of probabilities of each state ps_s. Each state is defined by a set of congested lines, so we can define a 0-1 matrix $\mathcal{K}$ whose element $\mathcal{K}(l,s)$ is 1 if line l is congested in state s and 0 otherwise. Denoting by **pl** and **ps** the vectors whose components are the lines' and states' probabilities, it can be written that:

$$\mathbf{pl} = \mathcal{K}\mathbf{ps}, \quad \mathbf{pl}^g = \mathcal{K}\mathbf{ps}^g, \forall g$$

Let us also define the complementary probabilities $ql_l = 1 - pl_l$. Finally, let us define $\mu_l = \max\left(\mu_{\min,l}, \mu_{\max,l}\right)$ and $\phi_l = \min\left(f_{\max,l} - f_l, f_l - f_{\min,l}\right)$.

Consistency requires that for every line, $pl_l \phi_l = 0$ and $ql_l \mu_l = 0$. So, if a line is surely congested, $pl_l = 1$ and $ql_l = 0$. Therefore, the slack variable $\phi_l = 0$ and the multiplier can take any value. In the case that the line is surely not congested, the multipliers $\mu_{\min,l}$ and $\mu_{\max,l}$ must be zero and the flow can take any value. Finally, if $0 < pl_l, ql_l < 1$, the flow must be equal to its bound and the multiplier must be zero. This is the critically

[2] After introducing outputs bounds.

congested case, where the previous section algorithm does not work. It also makes sense from a probabilistic point of view: if there is an slight uncertainty the flow must be equal to its bound or slightly below, and the multiplier must be zero or slightly positive. In the limit, the previous complementary conditions are obtained. Note that, being equivalent probabilities, the former consistency condition is equivalent to $pl_l^g \phi_l = 0$ and $ql_l^g \mu_l = 0, \forall g$.

IV. The enhanced algorithm: implementation

The proposed algorithm aims to compute a set of consistent probabilities ps_s. Note that problem (8) is parametrized by $\mathbf{ps}$ (as $\mathbf{ps}^g$ is a continuous function of $\mathbf{ps}$), so by solving (8) we can define the functions $\phi_l(\mathbf{ps})$ and $\mu_l(\mathbf{ps})$. These functions are continuous ones, as we deal with continuous modification of the quadratic term in the objective function of a QL program.

The proposed algorithm aims to solve the complementary problem:

$$\begin{array}{ccc} \mathbf{pl} = \mathcal{K}\mathbf{ps}, & \mathbf{ql} = \mathbf{1} - \mathbf{pl}, & \mathbf{1}^T\mathbf{ps} = 1, \\ pl_l \phi_l(\mathbf{ps}) = 0, & ql_l \mu_l(\mathbf{ps}) = 0 & \mathbf{ps} \geq \mathbf{0} \end{array} \quad (9)$$

where $\mathbf{1}$ and $\mathbf{0}$ are vectors whose components are all 1 and 0, respectively. It is possible to write this problem as a variational problem. Let us define

$$\begin{aligned} \mathbf{F} &= \left[\phi_1(\mathbf{ps}) \ldots \phi_L(\mathbf{ps}), N_\mu \mu_1(\mathbf{ps}) \ldots N_\mu \mu_L(\mathbf{ps}), 0 \ldots 0\right]^T \\ \mathbf{x} &= \left[pl_1 \ldots pl_L, ql_1 \ldots ql_L, ps_1 \ldots ps_S\right]^T \end{aligned}$$

being N_μ a scaling factor introduced by numerical reasons. Likewise, let us define the set

$$X = \left\{\mathbf{pl}, \mathbf{ql}, \mathbf{ps} \setminus \mathbf{pl} = \mathcal{K}\mathbf{ps}, \mathbf{ql} = \mathbf{1} - \mathbf{pl}, \mathbf{1}^T\mathbf{ps} = 1, \mathbf{p_s} \geq \mathbf{0}\right\}$$

Then, problem (9) can be restated as finding a vector $\mathbf{x}^* \in X$ satisfying

$$\mathbf{F}(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) \geq 0, \forall \mathbf{x} \in X$$

As F is continuous and X is closed and convex, it can be proven that a solution exists. To compute it, the projection method has been used [4]. Starting with any $\mathbf{x}^0 \in X$, iteratively updates $\mathbf{x}$ by using the formula

$$\mathbf{x}^{k+1} = \mathbb{P}_X\left[\mathbf{x}^k - \alpha \mathbf{F}\left(\mathbf{x}^k\right)\right]$$

where $\mathbb{P}_X$ denotes the orthogonal projection map onto X and α is a judiciously chosen positive stepsize.

The total number of network states is a very high one. So, only those that can possibly have non-negative probabilities are kept during the algorithm execution. Let us denote by $\mathcal{K}_\Omega$ the matrix $\mathcal{K}$ restricted to a given set of states $\Omega = \{s_1, s_2, \ldots\}$ (that is, only the columns corresponding to the states in Ω are preserved). Likewise, $\mathbf{F}_\Omega$, $\mathbf{x}_\Omega$ and X_Ω are defined.

Then, the following algorithm is proposed:

1. Apply the algorithm described in section 2. If the algorithm converges to a single state, a type I equilibrium has been computed, and the algorithm end. If the algorithm converges to a limit cycle, go to the following step.
2. Initialize Ω with the states in the limit cycle. Initialize $\mathbf{x}^0$ to any $\mathbf{x}_\Omega \in X_\Omega$. Set $k = 0$.
3. Use the probabilities $\mathbf{ps}$ in $\mathbf{x}^k$ to solve optimization problem (8). Then, check the solution state s^k. If $s^k \notin \Omega$, update $\Omega = \Omega \cup s^k$ and the related entities $\mathcal{K}_\Omega$, $\mathbf{x}_\Omega$, *et caetera*.
4. Update $\mathbf{x}_\Omega^{k+1} = \mathbb{P}_{X_\Omega}\left[\mathbf{x}_\Omega^k - \alpha \mathbf{F}_\Omega\left(\mathbf{x}_\Omega^k\right)\right]$.
5. Check convergence, and finish or go to 3.

V. Case example

The 6-buses network described in [5] and shown in figures 1 and 3 has been used. It is assumed that there are generators in buses 1, 2 and 3 belonging to different utilities. The cost functions are (cost in €, power in MW):

$$\begin{aligned} C_1(P_1) &= 600 + 6P_1 + 0.002P_1^2 \\ C_2(P_2) &= 220 + 7.3P_2 + 0.003P_2^2 \\ C_3(P_3) &= 100 + 8P_3 + 0.0042P_3^2 \end{aligned}$$

Buses 4, 5 and 6 are load buses. The demand functions are defined by (D_{0i} in MW, α_i in MW/€):

$$\begin{array}{lllllll} D_{04} & = & 114.96 & ; & \alpha_4 & = & 2.34 \\ D_{05} & = & 114.96 & ; & \alpha_5 & = & 2.34 \\ D_{06} & = & 114.98 & ; & \alpha_6 & = & 1.70 \end{array}$$

Remaining data are as in [5], but maximum flow through line 2-4, that is set to 35 MW. This case is studied in [3], showing that section II algorithm oscillates between two states. In state 1 there is no congested line, and in case 2 line 2-4 is congested.

If all the weight factors are set to $1 (w_s^g = 1, \forall s, g)$, the algorithm proposed in this paper converges to the solution shown in figure 1. Final state probabilities are $ps_1 = 0.85273$ and $ps_2 = 0.14727$. Note that line 2-4 is critically congested: the flow is the maximum flow and the associated multiplier is zero. In order to show that the computed solution is a Nash-Cournot equilibrium, figure 2 shows the change in the profit of each generator when each one changes unilaterally its output from the solution ones.

Another case is studied, with different weight factors. Specifically, $w_1^2 = 2$ and $w_2^3 = 2$, being all the remaining weight factors 1. Final state probabilities are $ps_1 = 0.77258$ and $ps_2 = 0.22742$. Results are shown in figures

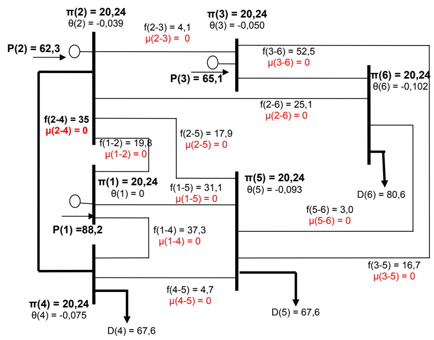

Fig. 1. Study network. Case 1.

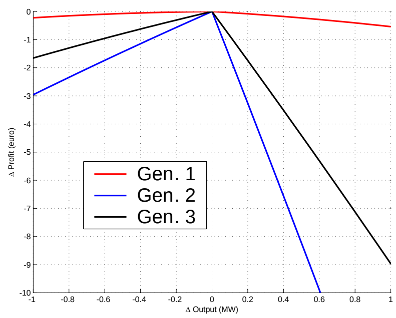

Fig. 2. Profit changes. Case 1.

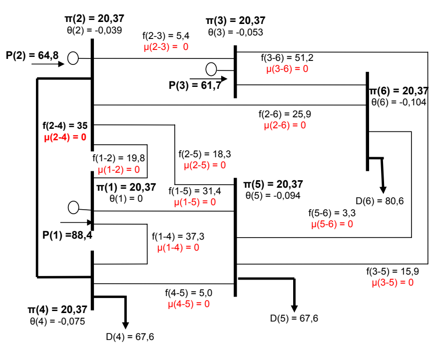

Fig. 3. Study network. Case 2.

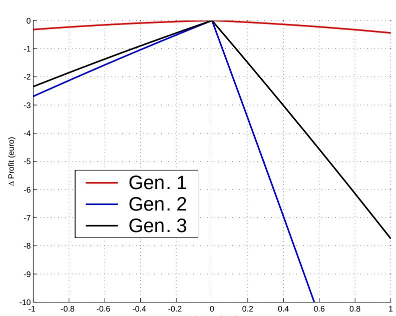

Fig. 4. Profit changes. Case 2.

3 and 4. Note that we have computed a different Nash-Cournot equilibrium.

Generally, an equilibria continuum can be computed by changing the weight factors values. Up to the best of author's knowledge, this sort of continuos equilibria have only been previously discussed in [6], albeit in a different way.

VI. Conclusion

A new algorithm to compute Nash-Cournot equilibria in critically congested networks has been proposed. The algorithm can be tuned to compute specific solutions in the generally continuous solution set. Results for a small test system are shown.

The preliminary results in this paper show the existence of a continuum of Nash-Cournot equilibria in critically congested systems. However, it should be kept in mind that the algorithm searches for a "stationary" point of the profit function. In other words, solution can show that one or more agents are at a minimum, rather than a maximum, of their profit functions.

VII. References

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