Cournot equilibrium computation on electricity networks

Julián Barquín

Miguel Vázquez

***Abstract*—Deregulated electricity markets are very often oligopolistic ones. Transmission constraints can significatively increase generators market power. Given the peculiar characteristics of electricity markets, new tools are required to quantify the generators market power. This paper presents such a tool, based on Cournot-like network analysis. The theoretical basis of the method as well as a study case are included.**

Keywords: **Power system economics, Interconnected power systems.**

I. Introduction

Electricity markets integration is seen by a number of regulators and analysts like an effective way of increasing competitive pressures in the sector. Actually, technical characteristics of the electricity made the sector prone to oligopolistic behaviour.

Traditional market power indexes are of limited use in electricity markets. A popular substitute have been the computation of Cournot-Nash equilibria. Although real competition is far more complex than Cournot competition, it is widely argued than Cournot equilibrium can be usually considered as the worst case of non-collusive competition. Therefore, it provides a bound to real behaviour, being quite useful from the regulatory point of view.

There are a number of models able to compute Cournot equilibrium in single-bus systems [1, 2, 3]. There are as well some models able to compute it in meshed systems [4, 5], although the computational effort required to study real-size systems is quite extensive.

In this paper we present a new algorithm to compute the Cournot equilibrium in meshed systems. The algorithm is an extension of the work in [6]. We feel that its computational efficiency and robustness are a significative improvement over the previous ones. A study case is also provided.

The sequel of this paper is organized as follows. Section 2 describes the proposed model, and section 3 the solution algorithm. A study case is shown in the next section. Finally, our conclusions are stated. Some mathematical proofs are collected in appendices.

J. Barquín and M. Vázquez are with Instituto de Investigación Tecnológica, Universidad Pontificia Comillas, C/ Alberto Aguilera 25, 28015 Madrid, Spain. Corresponding author is Dr. Barquín (e-mail: barquin@iit.upco.es).

II. The model

A single period, purely thermal model is proposed. It would not be difficult (albeit maybe very computationally consuming) to generalize the model to a multi-period one, including intertemporal constraints as the hydro ones, following the work in [6]. However, as the purpose of this study is to focus in network-related effects, a simplified single period model is appropriate.

A. General characteristics

The assumed model is that of a single period, purely thermal system. It is further considered

- n_g generators (utilities) that behave as Cournot oligopolists, i.e., set their units outputs (MW) in order to maximize their profit assuming known demand curves and fixed competitors' outputs.
- n_u thermal units. P_{gu} is the power output of unit u belonging to generator g. This unit is placed in bus $i(u)$ (each generator may have several units in several buses, and in each bus may be units belonging to several generators). The unit is characterized by a maximum output $P_{gu}^{\max}$, and a convex cost function $C_{gu} = C_{gu}(P_{gu})$. The minimum output is 0.
- A network made by n_{bus} buses and n_{lin} transmission lines. Each line is characterized by an admittance y_l and a maximum flow $f_l^{\max}$, so that the actual flow f_l fulfils $-f_l^{\max} \leq f_l \leq f_l^{\max}$. DC power flow equations are assumed, so that if line l goes from bus $i(l)$ to bus $j(l)$, the flow fulfils $f_l = y_l \left(\theta_{i(l)} - \theta_{j(l)}\right)$. θ_i is the voltage phase at bus i. No losses are considered
- A demand at each bus i which fulfils $D_i = D_{0i} - \alpha_i \pi_i$, being D_i the actual demand (MW), π_i the price (€/MW-h), and D_{0i} and α_i known constants.

The generators simultaneously submit bids for each one of their units. Each bid is a quantity bid, the amount of power to be produced. A central auctioneer receives the bids and decides the amount of demand to be consumed by each bus. This is done by maximizing the consumers utility subject to fulfil the network constraints.

B. Generators' behaviour

Each generator sets the output of is units in order to maximize its profit. Therefore, in equilibrium, marginal revenue and cost must be equal:

$$MR_{gu} = MC_{gu}$$

Marginal cost is just the derivative $\frac{dC_{gu}}{dP_{gu}}$. Marginal revenue has two components: the price $\pi_{i(gu)}$ at the bus where unit gu is located, and the decrease of the profit in each bus where generator u has units because of the decrease in the price because of the additional generation $\frac{d\pi_{i(u')}}{dP_{gu}} P_{gu'}$. Therefore, assuming that the unit is operating somewhere between its limits,

$$\frac{dC_{gu}}{dP_{gu}} = \pi_{i(u)} + \sum_{gu'} \frac{d\pi_{i(u')}}{dP_{gu}} P_{gu'}$$

In order to write this expression in a more easily manipulable form, let us introduce the following notation:

- $\mathbf{P}_g$ is a vector containing all the units belonging to generator g:

$$\mathbf{P}_g = \begin{bmatrix} P_{gu_1} \\ P_{gu_2} \\ \vdots \end{bmatrix}$$

- C_g is the total operating cost of generator g: $C_g = \sum_{gu} C_{gu}$.
- $\mathcal{C}_g$ is a 0-1 matrix which maps units into buses. That is, if element (i, j) of matrix $\mathcal{C}_g$ is 1, that means that unit gu_j is placed at bus i.
- $\boldsymbol{\pi}$ is the vector containing the nodal prices:

$$\boldsymbol{\pi} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \end{bmatrix}$$

- $\mathcal{S}$ is a matrix, whose element (i, j) is the minus change of the price in bus j when an additional MW is injected in bus i, assuming that generated outputs are hold constant (only demands are allowed to change). The demand variations are such that total demand utility (over all the buses) is maximized.

$$\mathcal{S}(i, j) = -\frac{\partial \pi_i}{\partial P_j}$$

By using the above symbols, equilibrium conditions for generator g can be written:

$$\nabla_{\mathbf{P}_g} C_g(\mathbf{P}_g) + \mathcal{C}_g^T \mathcal{S} \mathcal{C}_g \mathbf{P}_g = \mathcal{C}_g^T \boldsymbol{\pi}$$

where $\mathcal{C}_g^T$ is the transpose of $\mathcal{C}_g$.

C. Optimization approach

In equilibrium, the demand equations as well as the network equations must be fulfilled. Let me define the additional symbols:

- $\mathbf{D}$ is a vector containing the demands at each bus.
- $\boldsymbol{\theta}$ is the vector containing the phases at the different buses, but the bus 1 whose phase is set to 0.

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_2 \\ \theta_3 \\ \vdots \end{bmatrix}$$

- $\mathbf{f}$ a vector containing the flows. $\mathbf{f}^{\max}$ contains the limits.
- $\mathcal{F}$ is the matrix, obtained from the admittance data, relating flows and phases: $\mathbf{f} = \mathcal{F}\boldsymbol{\theta}$.
- $\mathcal{M}$ is the buses-lines incidence matrix ($\mathcal{M}(i, j) = 1$ if line j is leaving bus i, and -1 if it is arriving at bus i).

Therefore, the network equations can be written as:

$$\begin{aligned} \mathbf{D} + \mathcal{M}\mathbf{f} &= \sum_g \mathcal{C}_g \mathbf{P}_g \\ \mathbf{f} &= \mathcal{F}\boldsymbol{\theta} \\ -\mathbf{f}^{\max} \leq \mathbf{f} &\leq \mathbf{f}^{\max} \end{aligned}$$

The demand equations are

$$\frac{D_{0i} - D_i}{\alpha_i} = \pi_i$$

Let me define

- U_i is the bus i demand dis-utility:

$$U_i = \frac{1}{2\alpha_i} \left(D_{0i} - D_i\right)^2$$

- U is the total demand dis-utility $U = \sum_i U_i$.

Then, the demand equations can be written as:

$$\nabla_{\mathbf{D}} U(\mathbf{D}) = -\boldsymbol{\pi}$$

Putting all together, the equilibrium equations are:

$$\begin{aligned} \nabla_{\mathbf{P}_g} C_g(\mathbf{P}_g) + \mathcal{C}_g^T \mathcal{S} \mathcal{C}_g \mathbf{P}_g &= \mathcal{C}_g^T \boldsymbol{\pi} \\ \nabla_{\mathbf{D}} U(\mathbf{D}) &= -\boldsymbol{\pi} \\ \mathbf{D} + \mathcal{M}\mathbf{f} &= \sum_g \mathcal{C}_g \mathbf{P}_g \\ \mathbf{f} &= \mathcal{F}\boldsymbol{\theta} \\ -\mathbf{f}^{\max} \leq \mathbf{f} &\leq \mathbf{f}^{\max} \end{aligned}$$

Matrix $\mathcal{S}$ is positive symmetric (see below). *Assuming that it is given*, it is immediate to check that the above equations are the optimality conditions of the problem[1]:

$$\begin{array}{ll} \min_{\mathbf{P}_g,\mathbf{D},\mathbf{f},\boldsymbol{\theta}} & \sum_g C_g(\mathbf{P}_g) + \frac{1}{2}\mathbf{P}_g^T\mathcal{C}_g^T\mathcal{S}\mathcal{C}_g\mathbf{P}_g + U(\mathbf{D}) \\ \text{s.t.} & \left\{ \begin{array}{rcl} \mathbf{D}+\mathcal{M}\mathbf{f} & = & \sum_g \mathcal{C}_g\mathbf{P}_g \\ \mathbf{f} & = & \mathcal{F}\boldsymbol{\theta} \\ \mathbf{0} \leq & \mathbf{P}_g & \leq \mathbf{P}_g^{\max} \\ -\mathbf{f}^{\max} \leq & \mathbf{f} & \leq \mathbf{f}^{\max} \end{array} \right. \end{array} \tag{1}$$

The problem is a well defined one, as C_{gu} and U are convex functions and $\mathcal{S}$ a symmetric positive matrix (see appendices). Prices $\boldsymbol{\pi}$ are the multipliers of demand-generation constraints $\mathbf{D}+\mathcal{M}\mathbf{f} = \sum_g \mathcal{C}_g\mathbf{P}_g$.

Note that this problem includes the central auctioneer actions. In fact, if the solution of (1) is $\{\mathbf{P}_g^*, \mathbf{D}^*, \mathbf{f}^*, \boldsymbol{\theta}^*\}$, it is obvious that the solution with the additional constraint $\mathbf{P}_g = \mathbf{P}_g^*$ does not change. But if this last constraint is added, the unit outputs are constant, so they can be considered as the bids known by the central auctioneer; and the generators cost and additional quadratic term are constants as well, so that the objective function is just the demand dis-utility. So, this problem is just the auctioneer problem.

D. $\mathcal{S}$ computation

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In problem (1) it is assumed that $\mathcal{S}$ is given. But $\mathcal{S}$ depends on the system flows (specifically, on which lines are constrained). Therefore, an issue of consistency arises. The purpose of this subsection is to show how, from the nodal prices and knowledge of the constrained lines, matrix $\mathcal{S}$ can be computed.

$\mathcal{S}$ is the price sensitivity to a marginal increase in the injected power. Therefore, incremental changes $\delta\mathbf{D}$, $\delta\boldsymbol{\pi}$, $\delta\boldsymbol{\theta}$ and so on will be considered.

Firstly, network equations must be fulfilled.

$$\begin{array}{rcl} \delta\mathbf{D}+\mathcal{M}\delta\mathbf{f} & = & \delta\mathbf{P} \\ \delta\mathbf{f} & = & \mathcal{F}\boldsymbol{\delta\theta} \end{array}$$

$\delta\mathbf{P}$ is a vector of additional injections (units outputs are assumed constants). Then, in the constrained lines (those with flows at limits), $\delta f_l = 0$. Let me introduce a 0-1 matrix $\mathcal{E}$, with number of rows equal to the number of constrained lines and columns equal to the number of lines, such that the above conditions can be stated as

$$\mathcal{E}\delta\mathbf{f} = \mathbf{0}$$

On the other hand, it is assumed that the additional MW is shared by the demand (assigned by the central auctioneer) in order to maximize the total utility. But

$$\begin{aligned} U(\mathbf{D}+\delta\mathbf{D}) &= \sum_i \frac{1}{2\alpha_i}\left(D_{0i}-D_i-\delta D_i\right)^2 \\ &= \sum_i \frac{1}{2\alpha_i}\left(\delta D_i\right)^2 - \frac{D_{0i}-D_i}{\alpha_i}\delta D_i \\ &\quad + \frac{1}{2\alpha_i}\left(D_{0i}-D_i\right)^2 \\ &= \sum_i \frac{1}{2\alpha_i}\left(\delta D_i\right)^2 + \pi_i\delta D_i \\ &\quad + \frac{1}{2\alpha_i}\left(D_{0i}-D_i\right)^2 \end{aligned}$$

So, the demand increments are found by solving

$$\begin{array}{ll} \min_{\delta\mathbf{D}} & \boldsymbol{\pi}^T\delta\mathbf{D} + \frac{1}{2}\delta\mathbf{D}^T\mathcal{A}\delta\mathbf{D} \\ \text{s.t.} & \left\{ \begin{array}{rcl} \delta\mathbf{D}+\mathcal{M}\delta\mathbf{f} & = & \delta\mathbf{P} \\ \delta\mathbf{f} & = & \mathcal{F}\boldsymbol{\delta}\theta \\ \mathcal{E}\delta\mathbf{f} & = & \mathbf{0} \end{array} \right. \end{array}$$

where $\mathcal{A}$ is a diagonal matrix whose element $\mathcal{A}(i,i) = \frac{1}{\alpha_i}$.

By introducing multipliers $\boldsymbol{\lambda}_\mathbf{D}$, $\boldsymbol{\lambda}_\mathbf{f}$ and $\boldsymbol{\lambda}_\mathbf{s}$, the first order optimality conditions can be written:

$$\begin{bmatrix} \mathcal{A} & 0 & 0 & 0 & 0 & \mathcal{I} \\ 0 & 0 & 0 & \mathcal{F}^T & 0 & 0 \\ 0 & 0 & 0 & -\mathcal{I} & \mathcal{E}^T & \mathcal{M}^T \\ 0 & \mathcal{F} & -\mathcal{I} & 0 & 0 & 0 \\ 0 & 0 & \mathcal{E} & 0 & 0 & 0 \\ \mathcal{I} & 0 & \mathcal{M} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\mathbf{D} \\ \delta\boldsymbol{\theta} \\ \delta\mathbf{f} \\ \boldsymbol{\lambda}_\mathbf{f} \\ \boldsymbol{\lambda}_\mathbf{s} \\ \boldsymbol{\lambda}_\mathbf{D} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{\pi} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \delta\mathbf{P} \end{bmatrix}$$

where $\mathcal{I}$ is an identity matrix of the required dimension.

In order to compute $\mathcal{S}$, it is convenient to compute the matrix $\mathcal{S}_D$, whose element (i,j) is the change in the demand at bus i given a marginal power injection at bus j. It is clear that

$$\mathcal{S} = \mathcal{A}\mathcal{S}_D \tag{2}$$

as matrix $\mathcal{A}$ has the price "elasticities" as diagonal elements. On the other hand, it is also clear that matrix $\mathcal{S}_D$ can be computed by solving

$$\begin{bmatrix} \mathcal{A} & 0 & 0 & 0 & 0 & \mathcal{I} \\ 0 & 0 & 0 & \mathcal{F}^T & 0 & 0 \\ 0 & 0 & 0 & -\mathcal{I} & \mathcal{E}^T & \mathcal{M}^T \\ 0 & \mathcal{F} & -\mathcal{I} & 0 & 0 & 0 \\ 0 & 0 & \mathcal{E} & 0 & 0 & 0 \\ \mathcal{I} & 0 & \mathcal{M} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{S}_D \\ \vdots \end{bmatrix} = \begin{bmatrix} -\boldsymbol{\pi}\mathbf{1}^T \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathcal{I} \end{bmatrix} \tag{3}$$

[1]We have reintroduced at this point the bounds on generator outputs. It is also easy to check (albeit a bit cumbersome) that in that way the first order optimality conditions are the same equations than the equilibrium ones when limits are considered

where **1** is a column vector, all which elements are 1. The right hand side of this equation, as well as the big matrix, are of course known.

III. The algorithm

The proposed algorithm amounts essentially to iterate between equations (1), and (2) and (3). So, an initial constraints' state for each line is assumed, and equations (2) and (3) are solved to compute $\mathcal{S}$. Then, problem (1) is solved, and the lines' status (constrained on or off) assessed. If this state is the same one than the formerly assumed, the algorithm ends. If not, the new lines' status computed when solving (1) is plugged in (2) and (3), and the whole procedure iterated.

Specification of the lines' status can be done by assigning 0 to unconstrained lines and 1 to constrained ones. So $\{0, 0, 1, 0, ...\}$ denotes the state where the first, second and fourth lines are not constrained and the third line is. It is obvious that the lines' status is an element of the discrete set $\{0, 1\}^{n_{lin}}$.

So, each algorithm iteration can be understood as a map from a discrete set into itself. Therefore, only two outcomes are possible:

1. The algorithm converges to a single state in a finite number of iterations.
2. The algorithm converges to a limit cycle.

In the first case, the algorithm output is a Nash-Cournot equilibrium. In the second case, it can not be generally shown that there is not any Nash-Cournot equilibrium. On the other hand, there are systems that do not posses any Cournot equilibria; and we think possible confident that, given reasonable initial assumptions, converge to a limit cycle can usually means absence of such equilibria. More research is needed in this topic.

IV. Study case

The 6-buses network described in [7] and shown in figure 1 has been used. It is assumed that there are generators in buses 1, 2 and 3 belonging to different utilities. The cost functions are (cost in €, power in MW):

$$
\begin{aligned}
C_1(P_1) &= 600 + 6P_1 + 0.002P_1^2 \\
C_2(P_2) &= 220 + 7.3P_2 + 0.003P_2^2 \\
C_3(P_3) &= 100 + 8P_3 + 0.0042P_3^2
\end{aligned}
$$

Buses 4, 5 and 6 are load buses. The demand functions are defined by (D_{0i} in MW, α_i in MW^2/EUR):

$$
\begin{aligned}
D_{04} &= 114.96 \quad ; \quad \alpha_4 = 2.34 \\
D_{05} &= 114.96 \quad ; \quad \alpha_5 = 2.34 \\
D_{06} &= 114.98 \quad ; \quad \alpha_6 = 1.70
\end{aligned}
$$

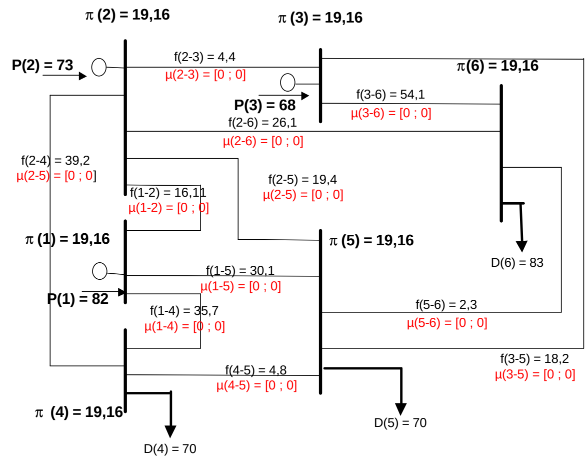

Fig. 1. Study network. Line 2-4 maximum flow: 40MW

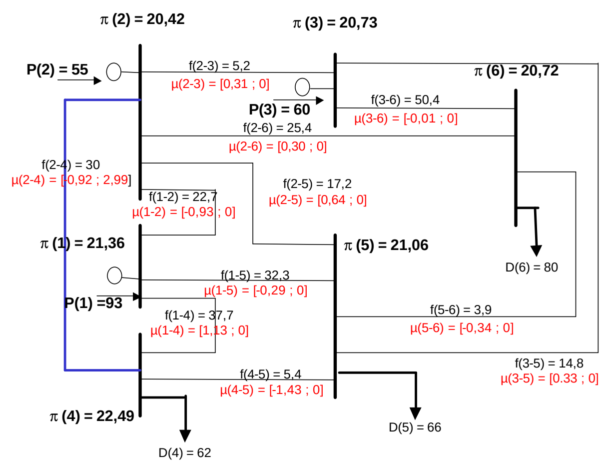

Fig. 2. Study network. Line 2-4 maximum flow: 30MW

Remaining data are as in [7]. Proposed algorithm converges to the results shown in figure 1. No line is constrained on, and prices $\boldsymbol{\pi}$ are uniform across the network. Symbol $\boldsymbol{\mu}$ denotes the multipliers pair

$$\boldsymbol{\mu} = [\boldsymbol{\mu}_f, \boldsymbol{\mu}_s]$$

where $\boldsymbol{\mu}_f$ are the multipliers of equation $\mathbf{f} = \mathcal{F}\boldsymbol{\theta}$, and $\boldsymbol{\mu}_s$ the greatest multiplier of constraint $-\mathbf{f}^{\max} \leq \mathbf{f} \leq \mathbf{f}^{\max}$; in problem (1).

If maximum power through line 2-4 is lowered from its base-case value of 40 MW to 30 MW a new solution, shown in figure 2 is computed. In the new solution only line 2-4 is constrained on, as shown by multipliers $\boldsymbol{\mu}_s$. Prices in each bus are different from each other, being its difference multipliers $\boldsymbol{\mu}_f$.

For maximum line 2-4 flows between 30 and 40 MW the algorithm can oscillate between two solutions, corre-

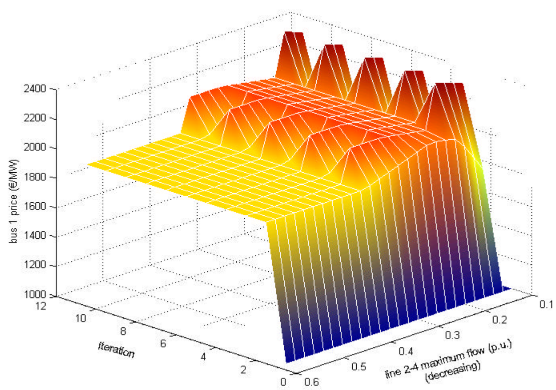

Fig. 3. Bus 1 prices.

sponding to have line 2-4 constrained on or constrained off. Bus 1 prices, in function of iteration number, are shown in figure 3. Therefore, in this simple system, no Cournot-Nash equilibrium can be computed. A similar situation is obtained when line 2-4 maximum flow is lowered below 20 MW, because of oscillations in the lines 2-5 and 2-6 status (constrained on or off).

V. References

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VI. Conclusions

A new tool to compute Nash-Cournot equilibria in power networks has been introduced. New theoretical results has been obtained when developping it. Absence of equilibria implies oscillations during the iterative procedure. The tool has been tested in a small power system.

VII. Acknowledgments

Help and remarks from Karsten Neuhoff and Gert Brunekreeft are gratefully acknowledge.

VIII. Appendices

A. Proof of the symmetry of $\mathcal{S}$

Let us define

$$\mathcal{B} = [0, 0, 0, 0, \mathcal{I}]$$

and

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & \mathcal{F}^T & 0 & 0 \\ 0 & 0 & -\mathcal{I} & \mathcal{E}^T & \mathcal{M}^T \\ \mathcal{F} & -\mathcal{I} & 0 & 0 & 0 \\ 0 & \mathcal{E} & 0 & 0 & 0 \\ 0 & \mathcal{M} & 0 & 0 & 0 \end{bmatrix}$$

Note that $\mathcal{C}$ is symmetric. So,

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^T & \mathcal{C} \end{bmatrix} \begin{bmatrix} \delta\mathbf{D} \\ \vdots \end{bmatrix} = \begin{bmatrix} -\boldsymbol{\pi} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \delta\mathbf{P} \end{bmatrix}$$

Therefore, after formal manipulation,

$$\left(\mathcal{A} - \mathcal{B}\mathcal{C}^{-1}\mathcal{B}^T\right)\delta\mathbf{D} = -\boldsymbol{\pi} - \mathcal{B}\mathcal{C}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \delta\mathbf{P} \end{bmatrix}$$

The problem here is that $\mathcal{C}$ is a singular matrix. This is immediate to check, as $\delta\mathbf{P} = \mathbf{0}$ implies $\delta\mathbf{D} = \mathbf{0}$ (if we are in equilibrium, no redistribution of demand can improve utility).

On the other hand, it is always possible to build a sequence $\mathcal{C}(\epsilon)$ of symmetric regular matrices whose limit as $\epsilon \to 0$ is $\mathcal{C}$. Let us define $\delta\bar{\mathbf{D}}(\epsilon, \delta\mathbf{P})$ as the solution of

$$\left(\mathcal{A}-\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\right)\delta\bar{\mathbf{D}}(\epsilon,\delta\mathbf{P}) = -\boldsymbol{\pi} - \mathcal{B}\mathcal{C}(\epsilon)^{-1}\begin{bmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\delta\mathbf{P}\end{bmatrix}$$

Let us define now the variable

$$\delta\mathbf{D}(\epsilon) = \delta\bar{\mathbf{D}}(\epsilon,\delta\mathbf{P}) - \delta\bar{\mathbf{D}}(\epsilon,\mathbf{0})$$

It is clear that, if $\delta\mathbf{D}$ is bounded (as it is, as it is solution of a strictly convex optimization problem), $\lim_{\epsilon\to 0}\delta\mathbf{D}(\epsilon) = \delta\mathbf{D}$. It is also clear that

$$\left(\mathcal{A}-\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\right)\delta\mathbf{D}(\epsilon,\delta\mathbf{P}) = -\mathcal{B}\mathcal{C}(\epsilon)^{-1}\begin{bmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\delta\mathbf{P}\end{bmatrix}$$

So, the matrix $\mathcal{S}_D(\epsilon)$ (sensitivity matrix relating increment $\delta\mathbf{D}_i(\epsilon)$ to power injection at bus j) fulfils

$$\begin{aligned}\left(\mathcal{A}-\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\right)\mathcal{S}_D(\epsilon) &= -\mathcal{B}\mathcal{C}(\epsilon)^{-1}\begin{bmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathcal{I}\end{bmatrix}\\ &= -\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\end{aligned}$$

So,

$$\begin{aligned}\mathcal{S}_D(\epsilon) &= -\left(\mathcal{A}-\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\right)^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\\ &= -\left(\mathcal{I}-\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\right)^{-1}\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\\ &= -\Big[\mathcal{I}+\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T+\\ &\quad\left(\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\right)^2+\ldots\Big]\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\\ &= -\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T-\\ &\quad\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T-\\ &\quad\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T+\ldots\end{aligned}$$

So,

$$\begin{aligned}\mathcal{S}(\epsilon) &= \mathcal{A}\mathcal{S}_D(\epsilon)\\ &= -\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T-\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T-\\ &\quad\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T+\ldots\end{aligned}$$

which is a clearly symmetric matrix. Taking the limit $\epsilon\to 0$, it is shown that $\mathcal{S}$ is symmetric.

There is a last loophole, to prove that the series is convergent. That amounts to the convergence of

$$\left(\mathcal{I}-\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T\right)^{-1}$$

which converges if the spectral radius of $\mathcal{A}^{-1}\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T$ is less than 1. As $\mathcal{A}$ is a positive diagonal matrix, that can be made sure by having $\mathcal{A}$ big enough, which can be made by choosing a suitable monetary unit.

B. Proof that $\mathcal{S}$ is positive

Let us consider a particular $\mathcal{C}(\epsilon)$, i.e.

$$\mathcal{C}(\epsilon) = \epsilon\mathcal{I}+\mathcal{C}$$

This perturbation moves the whole $\mathcal{C}$ spectrum by ϵ. As the spectrum is a discrete set, it is clear that there is a ϵ_1 such that for any ϵ in the interval $(0,\epsilon_1)$, matrix $\mathcal{C}(\epsilon)$ is not singular. We will restrict ourselves in the sequel to this interval.

Now, let us partition $\mathcal{C}(\epsilon)$ as

$$\mathcal{C}(\epsilon) = \begin{bmatrix}\epsilon & 0 & \mathcal{F}^T & 0 & 0\\ 0 & \epsilon & -\mathcal{I} & \mathcal{E}^T & \mathcal{M}^T\\ \mathcal{F} & -\mathcal{I} & \epsilon & 0 & 0\\ 0 & \mathcal{E} & 0 & \epsilon & 0\\ 0 & \mathcal{M} & 0 & 0 & \epsilon\end{bmatrix} = \begin{bmatrix}\epsilon\mathcal{I} & \mathcal{C}_r^T\\ \mathcal{C}_r & \epsilon\mathcal{I}\end{bmatrix}$$

where

$$\mathcal{C}_r = \begin{bmatrix}\mathcal{F} & -\mathcal{I}\\ 0 & \mathcal{E}\\ 0 & \mathcal{M}\end{bmatrix}$$

Now, it is easy to check

$$\mathcal{C}(\epsilon)^{-1} = \begin{bmatrix}\epsilon\mathcal{I} & -\mathcal{C}_r^T\\ -\mathcal{C}_r & \epsilon\mathcal{I}\end{bmatrix}\begin{bmatrix}\left(\epsilon^2\mathcal{I}-\mathcal{C}_r^T\mathcal{C}_r\right)^{-1} & 0\\ 0 & \left(\epsilon^2\mathcal{I}-\mathcal{C}_r\mathcal{C}_r^T\right)^{-1}\end{bmatrix}$$

On the other hand, let us write matrix $\mathcal{B}$ as

$$\mathcal{B} = [0,0,0,0,\mathcal{I}] = [0,\mathcal{J}]$$

with

$$\mathcal{J} = [0,\mathcal{I}]$$

So,

$$\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T = \epsilon\mathcal{J}\left(\epsilon^2\mathcal{I}-\mathcal{C}_r^T\mathcal{C}_r\right)^{-1}\mathcal{J}^T \qquad (4)$$

Now, it is immediate that

$$\mathcal{C}_r^T\mathcal{C}_r = \begin{bmatrix}\mathcal{F}^T\mathcal{F} & -\mathcal{F}^T\\ -\mathcal{F} & \mathcal{I}+\mathcal{E}^T\mathcal{E}+\mathcal{M}^T\mathcal{M}\end{bmatrix}$$

$\mathcal{C}_r^T\mathcal{C}_r$ is a positive matrix. To check that, let us consider

$$\left[\mathbf{v}_{\boldsymbol{\theta}}^T, \mathbf{v}_{\mathbf{f}}^T\right] \mathcal{C}_r^T \mathcal{C}_r \left[\begin{array}{c} \mathbf{v}_{\boldsymbol{\theta}} \\ \mathbf{v}_{\mathbf{f}} \end{array}\right] = \mathbf{v}_{\boldsymbol{\theta}}^T \mathcal{F}^T \mathcal{F} \mathbf{v}_{\boldsymbol{\theta}} + \mathbf{v}_{\mathbf{f}}^T \left(\mathcal{I} + \mathcal{E}^T\mathcal{E} + \mathcal{M}^T\mathcal{M}\right) \mathbf{v}_{\mathbf{f}} - 2\mathbf{v}_{\boldsymbol{\theta}}^T \mathcal{F}^T \mathbf{v}_{\mathbf{f}} = \left(\mathbf{v}_{\boldsymbol{\theta}}^T \mathcal{F}^T - \mathbf{v}_{\mathbf{f}}^T\right) \left(\mathcal{F}\mathbf{v}_{\boldsymbol{\theta}} - \mathbf{v}_{\mathbf{f}}\right) + \mathbf{v}_{\mathbf{f}}^T \left(\mathcal{E}^T\mathcal{E} + \mathcal{M}^T\mathcal{M}\right) \mathbf{v}_{\mathbf{f}}$$

It is obvious that this expression is greater than or equal to zero. Assume that it is zero. Then $\mathcal{M}\mathbf{v}_{\mathbf{f}} = \mathbf{0}$. That means that it is possible to consider $\mathbf{v}_{\mathbf{f}}$ as a set of flows such that the net power injection $\sum_g \mathcal{C}_g \mathbf{P}_g - \mathbf{D}$ in each bus is zero, so it is purely rotational set of flows. It must hold as well that $\mathcal{F}\mathbf{v}_{\boldsymbol{\theta}} - \mathbf{v}_{\mathbf{f}} = \mathbf{0}$, which means that $\mathbf{v}_{\boldsymbol{\theta}}$ can be consider as the set of phases that induces these flows. But the only rotational flow that a set of phases can induce is the zero flows $\mathbf{v}_{\mathbf{f}} = \mathbf{0}$, at the phases that induce these flows must be equal in every bus, so $\mathbf{v}_{\boldsymbol{\theta}} = \mathbf{0}$ because $\theta_1 = 0$. Then, I have proven $\mathcal{C}_r^T\mathcal{C}_r > 0$.

Let be $\epsilon_2 > 0$ the smallest $\mathcal{C}_r^T\mathcal{C}_r$ eigenvalue. Then, if $\epsilon < \min(\sqrt{0.5\epsilon_2}, \epsilon_1)$, matrix $\left(\epsilon^2\mathcal{I} - \mathcal{C}_r^T\mathcal{C}_r\right)$ must be definite negative, and therefore matrix $\mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T$ in equation ((4)) (which is the restriction to a lower dimensional subspace) as well.

Let me call $\mathcal{C}_{\mathcal{B}}(\epsilon) = \mathcal{B}\mathcal{C}(\epsilon)^{-1}\mathcal{B}^T$. $\mathcal{C}_{\mathcal{B}}(\epsilon)$ spectrum is bounded. If $\mathcal{C}_r^T\mathcal{C}_r$ spectrum is in the interval $[\epsilon_2, E_2]$, $\mathcal{C}_{\mathcal{B}}(\epsilon)$ must be in $[-E_2, -0.5\epsilon_2] = [-K, -k]$, because $0.5\epsilon_2$ is the greatest perturbation that $\epsilon^2\mathcal{I}$ can induce in $-\mathcal{C}_r^T\mathcal{C}_r$. Notice that $K, k > 0$ are two constants independent of ϵ.

Now, $-K\mathcal{I} < \mathcal{C}_{\mathcal{B}}(\epsilon) < -k\mathcal{I}$, so $-k^{-1}\mathcal{I} < \mathcal{C}_{\mathcal{B}}(\epsilon)^{-1} < K^{-1}\mathcal{I}$. On the other hand, $\min_i(\alpha_i)\mathcal{I} < \mathcal{A}^{-1} < \max_i(\alpha_i)\mathcal{I}$. Then

$$\left(\min_i(\alpha_i) + K^{-1}\right)\mathcal{I} < \mathcal{A}^{-1} - \mathcal{C}_{\mathcal{B}}(\epsilon)^{-1} < \left(\max_i(\alpha_i) + k^{-1}\right)\mathcal{I}$$

And, therefore

$$\left(\mathcal{A}^{-1} - \mathcal{C}_{\mathcal{B}}(\epsilon)^{-1}\right)^{-1} > \left(\max_i(\alpha_i) + k^{-1}\right)^{-1}\mathcal{I} = \bar{k}\mathcal{I} > 0$$

But,

$$\begin{aligned}
\left(\mathcal{A}^{-1} - \mathcal{C}_{\mathcal{B}}(\epsilon)^{-1}\right)^{-1} &= \mathcal{C}_{\mathcal{B}}(\epsilon)\left(\mathcal{A}^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon) - \mathcal{I}\right)^{-1} \\
&= -\mathcal{C}_{\mathcal{B}}(\epsilon)\left(\mathcal{I} - \mathcal{A}^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon)\right)^{-1} \\
&= -\mathcal{C}_{\mathcal{B}}(\epsilon)\left(\mathcal{I} + \mathcal{A}^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon) + \right. \\
&\quad \left. \mathcal{A}^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon)\mathcal{A}^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon) + \ldots\right) \\
&= -\mathcal{C}_{\mathcal{B}}(\epsilon) - \mathcal{C}_{\mathcal{B}}(\epsilon)\mathcal{A}^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon) - \\
&\quad \mathcal{C}_{\mathcal{B}}(\epsilon)\mathcal{A}^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon)\mathcal{A}^{-1}\mathcal{C}_{\mathcal{B}}(\epsilon) - \ldots \\
&= \mathcal{S}(\epsilon)
\end{aligned}$$

Therefore, $\mathcal{S}(\epsilon) > \bar{k}\mathcal{I}$ assuming that the series converges. And it does in the same hypothesis that in the previous appendix. As $\bar{k}$ is independent of ϵ, it must converge as $\epsilon \to 0$ to a positive matrix. Therefore $\mathcal{S} > 0$.

IX. Biographies

Julián Barquín (M' 94) received the Ingeniero Industrial Degree and the Doctor Ingeniero Industrial Degree from the Universidad Pontificia Comillas, Madrid, in 1988 and 1993 respectively; and the Licenciado en Ciencias Físicas Degree in 1994. He belongs to the research staff at the Instituto de Investigación Tecnolgica. His present interests include control, operation and planning of power systems.

Miguel Vázquez is about to obtain the Ingeniero Industrial Degree from the Universidad Politcnica de Madrid. He is interested in power systems and markets.